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The Existence of Global Attractor for a Strongly Continuous Semigroup in Metric Space

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ABSTRACT

The behavior of dynamic systems in long term dynamics can be described by the global attractor. Let V be a metric space, the global attractor is a nonempty, compact, and invariant set of A subset V which attracts every bounded subset of V. This paper proves three necessary and sufficient conditions that guarantee the existence of the global attractor for a strongly continuous semigroup in metric space: the semigroup is point dissipative, asymptotically smooth, and keeps orbit of bounded set bounded. The existence of global attractor is proved by using ω-limit compactness of the semigroup.

Key Words: global attractor, strongly continuous semigroup.

1. Introduction

Many mathematical physics problems can be put into the perspective of infinite dimensional systems and that system can be described by a strongly continuous semigroup in proper function spaces. One important object to describe the long time dynamics of the infinite dimensional system is the global attractor. In the past forty years, some mathematicians have successfully applied some theory in dynamical systems, real analysis, and functional analysis to show the existence of the global attractor. In lecture notes [1] and [2], a dynamical system is defined as a family of operator semigroups $\{T_t\}$ which map a metric space V to itself and are strongly continuous. That semigroup can describe asymptotic behavior of one dynamical system for a long time.

The global attractor is a basic concept and tool to study asymptotic behaviors of solutions of nonlinear evolution equations. A global attractor is an invariant compact set which absorbs all of the bounded sets as time goes to infinity. This theory has been investigated by several authors (cf. Cholewa and Dlotko [1], Ladyzhenskaya [3], Lan dan Shi [4]) and some of techniques seems to be useful for our problem.

The object of this paper is to show the existence of the global attractor for the strongly continuous semigroup in metric space. In [5], some techniques introduced to show the existence of the global attractor in abstract parabolic problem. Such a measure of noncompactness of the semigroup was used by Ma, Wang and Zhong [6]. But here, we show the existence of the global attractor by using ω-limit compactness of the semigroup.

2. Theory

Based on [5], [6], [7], some definitions in real analysis, functional analysis and dynamical systems are defined as follows.

Definition 2.1. *Let V be a metric space. The evolution of dynamical system in V is described by a family of operators $\{T(t)\}$ of maps V into itself. The family is called a C^0 semigroup if it satisfies*

(i) T_0 *is the identity map on* V,

(ii) $T_{(t+s)} = T_t \circ T_s$ *for every* $t, s \geq 0$,

(iii) *The function* $T: [0,\infty) \times V \to V$

$$\ni (t,x) \to T_t(x) \in V$$

is continuous at each point $(t,x) \in [0,\infty) \times V$.

Definition 2.2. *Let set K in metric space V. The set K is said to be compact if for every open K open bounded set subset in K.*

Definition 2.3. *Let V, Y be metric, map $T: V \to Y$ is continuous at $v_0 \in V$ if given any $\varepsilon > 0$ there exist $\delta > 0$ such that*

$$d'\big(T(v), T(v_0)\big) < \varepsilon \quad with \quad d(v, v_0) < \delta.$$

By geometrical views, let $B_Y(T(v_0), \varepsilon)$ open ball in $T(v_0)$ with radius ε, and $B_V(v_0, \delta)$ open ball at v_0 with radius δ, then

$$T(v) \in B_Y(T(v_0), \varepsilon) \ with \ v \in B_V(v_0, \delta)$$

or

$$T\big(B_V(v_0, \delta)\big) \subseteq B_Y(T(v_0), \varepsilon).$$

Theorem 2.1 *Let A, B be metric spaces and $f: A \to B$. If $\{S_i\}$ subset A and $i, n \in \mathbb{N}$, then*

$$f\left(\bigcup_{i=1}^{n} S_i\right) = \bigcup_{i=1}^{n} f(S_i).$$

Definition 2.4. *Let V be a metric space. Given sets $W_1, W_2 \subset V$, W_2 $\{T_t\}$ absorbed by W_1 if there exist $t_0 \geq 0$ such that*

$$T_t W_2 \subset W_1 \ for\ every \ t \geq t_0.$$

Definition 2.5. *Let V is a metric space. A Global attractor is a non empty, compact and $\{T_t\}$ −invariant set $\mathcal{A} \subset V$ which is attract every bounded set.*

Definition 2.6. *The semigrup $\{T_t\}$ is called point dissipative if and only if there exixst a non empty , bounded set $W_0 \subset V$ which attract every point in V. In other words, if $W_0 \subset V$ attracts every point in V, then $\forall_{v \in (V,d)}$ there exist an open neighborhood $\mathcal{N}_{W_0}$ of W_0 which absorb $v, \forall_{v\in(V,d)} \exists_{t_0 \geq 0} \forall_{t \geq t_0} T_t(v) \subset \mathcal{N}_{W_0}$.*

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Definition 2.7. *The semigroup* $\{T_t\}$ *is called asymptotically smooth if and only if there exists a non empty, closed, bounded, positively invariant set* $W_0 \subset V$ *contains a non empty, compact subset C which attracts* W_0.

Theorem 2.2. *Let* $\{T_t\}$ *be* C^0 *semigroup in metric space* V. *If* $\{T_t\}$ *is asimtotically smooth, B a non empty subset* V, *and for some* $t_B \geq 0$, *set* $\bigcup_{s\geq t_B} T_s(B)$ *bounded, then* $\omega(B)$*is a non empty, compact, and invariant set, moreover* $\omega(B)$ *attract* B.

3. Result and Discussion

In this section we give the existence of the global attractor for strongly continuous semigroup in metric space *V* describe in Theorem 3.1. below,

Theorem 3.1. *Let* $\{T_t\}$ *be a* C^0 *semigroup in metric space* V. *If* $\{T_t\}$ *is point disipative, asymptotically smooth and keep orbit of bounded set bounded then* $\{T_t\}$ *has a global attractor in* V.

Proof. The proof proceeds in two steps.

Step 1.

We shall first show the existence of a bounded set $\mathcal{O} \subset V$ such that for every bounded closed set $\mathcal{C} \subset V$ there is a neighborhood $\mathcal{N}_{\mathcal{C}}$ which is absorbed by $\mathcal{O}$.

Since semigroup $\{T_t\}$ is point dissipative, from Definition 2.6 there exist a non empty and bounded set $W_0 \subset V$ which attracts every point in V, therefore $\forall_{v\in V}$ every open neighborhood $\mathcal{N}_{W_0}$ from W_0 *absorbs* v. We can say that $\mathcal{N}_{W_0}$ is an absorbing point of V. From Definition 2.6 $\mathcal{N}_{W_0}$ *absorbs* every point v if $\forall_{v\in V}\exists_{t_0\geq 0}\forall_{t\geq t_0}$ such that $T_t(v) \subset \mathcal{N}_{W_0}$. (3.1)

Since $\{T_t\}$ is a C^0 semigroup in metric space V, from Definition 2.3 $\{T_t\}$ continuous in every point in V, hence $\forall_{v\in V}\forall_{\varepsilon>0}\exists_{\delta>0}$ such that $T_t(B_V(v,\delta)) \subset B_V(T_t(v),\varepsilon)$.

Because $\mathcal{N}_{W_0}$ is an absorbing point in V, then $\forall_{v\in(V,d)}\exists_{t_0\geq 0}\forall_{t\geq t_0} T_t(v) \subset \mathcal{N}_{W_0}$, so for $B_V(T_t(v),\varepsilon) \subset \mathcal{N}_{W_0}$, we can choose $B_V(v,\delta)$ such that $T_t(B_V(v,\delta)) \subset \mathcal{N}_{W_0}$, and we have $\forall_{v\in V}\exists_{B_{V(v,\delta)}}\exists_{t_0\geq 0}$ such that

$T_t(B_V(v,\delta)) \subset \mathcal{N}_{W_0}$. (3.2)

Let us choose $t_{\mathcal{N}_{W_0}} \geq 0$ such that

$$\mathcal{O} := \bigcup_{t\geq t_{\mathcal{N}_{W_0}}} T_t(\mathcal{N}_{W_0})$$

bounded.

Since $\{T_t\}$ is a strongly continuous semigroup in V, then $\mathcal{O}$ invariant positive, we obtain,

$$T_{t'}(\mathcal{O}) \subset \mathcal{O}, t' \geq 0.$$

Furthermore,

$$T_{t'}(\mathcal{O}) = T_{t'}\left(\bigcup_{t\geq t_{\mathcal{N}_{W_0}}} T_t(\mathcal{N}_{W_0})\right), t' \geq 0.$$

From Theorem 2.1. there follows

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$$T_{t'}(\mathcal{O}) = T_{t'}\left(\bigcup_{t\geq t_{\mathcal{N}W_0}} T_t(\mathcal{N}_{W_0})\right) = \bigcup_{t\geq t_{\mathcal{N}W_0}} T_{t'}(T_t(\mathcal{N}_{W_0})).$$
(3.3)

According assumptions that $\{T_{t'}\}$ and $\{T_t\}$ are strongly continuous semigroup and from Definition 2.1 Part (ii) we have

$$\bigcup_{t\geq t_{\mathcal{N}W_0}} T_{t'}(T_t(\mathcal{N}_{W_0})) = \bigcup_{t\geq t_{\mathcal{N}W_0}} T_{t'+t}(\mathcal{N}_{W_0}).$$

Next from

$$\bigcup_{t\geq t_{\mathcal{N}W_0}} T_{t'+t}(\mathcal{N}_{W_0}) = T_{t'+t_{\mathcal{N}W_0}}(\mathcal{N}_{W_0}) \cup T_{t'+t_{\mathcal{N}W_0}+1}(\mathcal{N}_{W_0}) \cup \dots .$$
(3.4)

And because $t' \geq 0$

$$\bigcup_{t\geq t_{\mathcal{N}W_0}} T_t(\mathcal{N}_{W_0}) = T_{t_{\mathcal{N}W_0}}(\mathcal{N}_{W_0}) \cup T_{t_{\mathcal{N}W_0}+1}(\mathcal{N}_{W_0}) \cup \dots \cup T_{t'+t_{\mathcal{N}W_0}}(\mathcal{N}_{W_0}) \cup T_{t'+t_{\mathcal{N}W_0}+1}(\mathcal{N}_{W_0}) \cup \dots$$

$$= T_{t_{\mathcal{N}W_0}}(\mathcal{N}_{W_0}) \cup T_{t_{\mathcal{N}W_0}+1}(\mathcal{N}_{W_0}) \cup \dots \cup \bigcup_{t\geq t_{\mathcal{N}W_0}} T_{t'+t}(\mathcal{N}_{W_0}) \cup \dots .$$
(3.5)

Therefore from Equations (3.3) , (3.4) and (3.5) we obtain

$$T_{t'}(\mathcal{O}) = \bigcup_{t\geq t_{\mathcal{N}W_0}} T_{t'+t}(\mathcal{N}_{W_0}) \subset \bigcup_{t\geq t_{\mathcal{N}W_0}} T_t(\mathcal{N}_{W_0}) = \mathcal{O}.$$

So, $T_{t'}(\mathcal{O}) \subset \mathcal{O}$, or $\mathcal{O}$ positive invariant.

Next, because $\mathcal{N}_{W_0}$ *absorbs* every point in V,

$$\forall_{v\in V}\exists_{t_0\geq 0} \text{ such that } T_t(v) \subset \mathcal{N}_{W_0}\ , t \geq t_0$$

Then there is $t_0 \geq 0$ such that

$$T_{t_{\mathcal{N}W_0}}(T_t(v)) \subset T_{t_{\mathcal{N}W_0}}(\mathcal{N}_{W_0}) \subset \bigcup_{t\geq t_{\mathcal{N}W_0}} T_t(\mathcal{N}_{W_0}) = \mathcal{O}, \qquad t_{\mathcal{N}W_0} \geq t_0.$$

Hence $\mathcal{O}$ *absorbs* every point in V.

According to Definition 2.1 Part (ii), $\{T_t\}$ is strongly continuous, hence

$$T_{t_{\mathcal{N}W_0}}(T_t(v)) = T_{t_{\mathcal{N}W_0}+t}(v).$$

With the aid of Equation (3.2) and Definition 2.1 part (ii) also with the fact that $\mathcal{O}$ is the absorbing point in V, $\forall_{v\in V}\exists_{t_v=t_{\mathcal{N}W_0}+t\geq 0}\exists_{B_V(v,\delta)}$ such that $T_t(B_V(v,\delta)) \subset \mathcal{O}, t \geq t_v$. (3.6)

Consider any compact set $C \subset V$, certainly from Definition 2.2,

$$C \subset \bigcup_{i=1}^{n} B_V(v_i,\delta_i)$$

$$C \subset B_V(v_1,\delta_1) \cup B_V(v_2,\delta_2) \cup \dots \cup B_V(v_n,\delta_n) =: \mathcal{N}_C$$
(3.7)

For some $n \in \mathbb{N}$, $v_1, v_2, \dots, v_n \in C$ and because T_t continuous, then

$$T_t(C) \subset T_t(\mathcal{N}_C) = T_t \bigcup_{i=1}^{n} B_V(v_i,\delta_i) = \bigcup_{i=1}^{n} T_t(B_V(v_i,\delta_i)).$$

According to Equation (3.6) we obtain

$$T_t(B_V(v_1,\delta_1)) \subset \mathcal{O}\ , \text{for } t \geq t_{v_1}$$

$$T_t(B_V(v_2,\delta_2)) \subset \mathcal{O}\ , \text{for } t \geq t_{v_2}$$

$$\vdots$$

$$T_t(B_V(v_n, \delta_n)) \subset \mathcal{O} \text{ , for } t \geq t_{v_n}.$$

We have for $t \geq \max\{t_{v_1}, t_{v_2}, \dots, t_{v_n}\} \geq 0$ that $T_t(\mathcal{N}_C) \subset \mathcal{O}$.
So we can show that there exists a set $\mathcal{O} \subset V$ such that every compact set $\mathcal{C} \subset V$ has a neighborhood $\mathcal{N}_\mathcal{C}$ which is absorbed by set $\mathcal{O}$.

Step 2

We show that there is a compact, $\{T_t\}$ invariant set $\mathcal{A} \subset V$which attract every bounded set in V.
Let $B \subset V$ be a non empty bounded.set. Since

(i) $\{T_t\}$ is asymptotically smooth and
(ii) From property of orbit of bounded set keep bounded we have

$$\gamma(B) = \bigcup_{s \geq 0} T_s(B)$$

is bounded, therefore for some $t_B \geq 0$, set

$$\bigcup_{s \geq t_B} T_s(B)$$

is bounded too, then from Theorem 2.1 we can conclude according to Theorem 2.2 that $\omega(B)$ compact and attract B, i.e.

$$\forall_{\mathcal{N}_{\omega(B)}} \exists_{t_B \geq 0} \; T_{t_1} B \subset \mathcal{N}_{\omega(B)} \text{ for } t_1 \geq t_B \qquad (3.8)$$

As shown in step I, there is some neighborhood of compact set $(\omega(B))$; $(\mathcal{N}_{\omega(B)})$; which absorbed by $\mathcal{O}$, i.e.

$$\exists_{t_0} T_{t_3}(\mathcal{N}_{\omega_B}) \subset \mathcal{O} \text{ for k } t_3 \geq t_0$$

Take $t_2 = t_3 + t_1$,

$$T_{t_2}(B) = T_{t_3+t_1}(B)$$
$$= T_{t_3}\left(T_{t_1}(B)\right) \subset T_{t_3}(\mathcal{N}_{\omega(B)}) \subset \mathcal{O}.$$

We obtain, there is $\tau_B = t_B + t_0$ such that

$$\forall_{B \subset V} \exists_{\tau_B \geq 0} \; T_{t_2}(B) \subset \mathcal{O} \text{ for } t_2 \geq \tau_B. \qquad (3.9)$$

Let, $\mathcal{A} := \omega(\mathcal{O})$. Because

(i) $\{\{T_t\}$ is asymptotically smooth and
(ii) Set $\mathcal{O}$ bounded and positive invariant such that $T_{t_2}(\mathcal{O}) \subset \mathcal{O}$, hence

$$\exists_{\tau_B \geq 0} \cup_{t_2 \geq \tau_B} T_{t_2}(\mathcal{O}) \text{ is bounded.}$$

From Theorem 2.2, $\mathcal{A} := \omega(\mathcal{O})$ is a non empty, compact set, moreover, $\mathcal{A}$ *attracts* $\mathcal{O}$.
Now we will show that $\mathcal{A}$ attracts every bounded set in V.

$\mathcal{A}$ attracts $\mathcal{O}$ means $\forall_{\mathcal{N}_\mathcal{A}} \exists_{t_0 \geq 0} T_t(\mathcal{O}) \subset \mathcal{N}_\mathcal{A}$ for every $t \geq t_0$.

From Equation (3.9), let $t_4 = t + t_2$

$$T_{t_4}(B) = T_{t+t_2}(B) = T_t\left(T_{t_2}(B)\right) \subset T_t(\mathcal{O}) \subset \mathcal{N}_\mathcal{A}$$

there is $\tau_2 = t_o + \tau_B$ such that

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$$\forall_{B\subset V}\exists_{\tau_2=t_0+\tau_B\geq 0}\, T_{t_4}(B)\subset \mathcal{N}_{\mathcal{A}} \text{ for every } t_4\geq \tau_2.$$

Hence $\mathcal{A}$ attract every bounded set in V. The proof is complete.■.

5. Conclusion

Let $\{T_t\}$ be a C^0 semigroup in metric space V. There exist a global attractor in V if and only if the semigroup $\{T_t\}$ is point disipative, asymptotically smooth and keep orbit of bounded set bounded.

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