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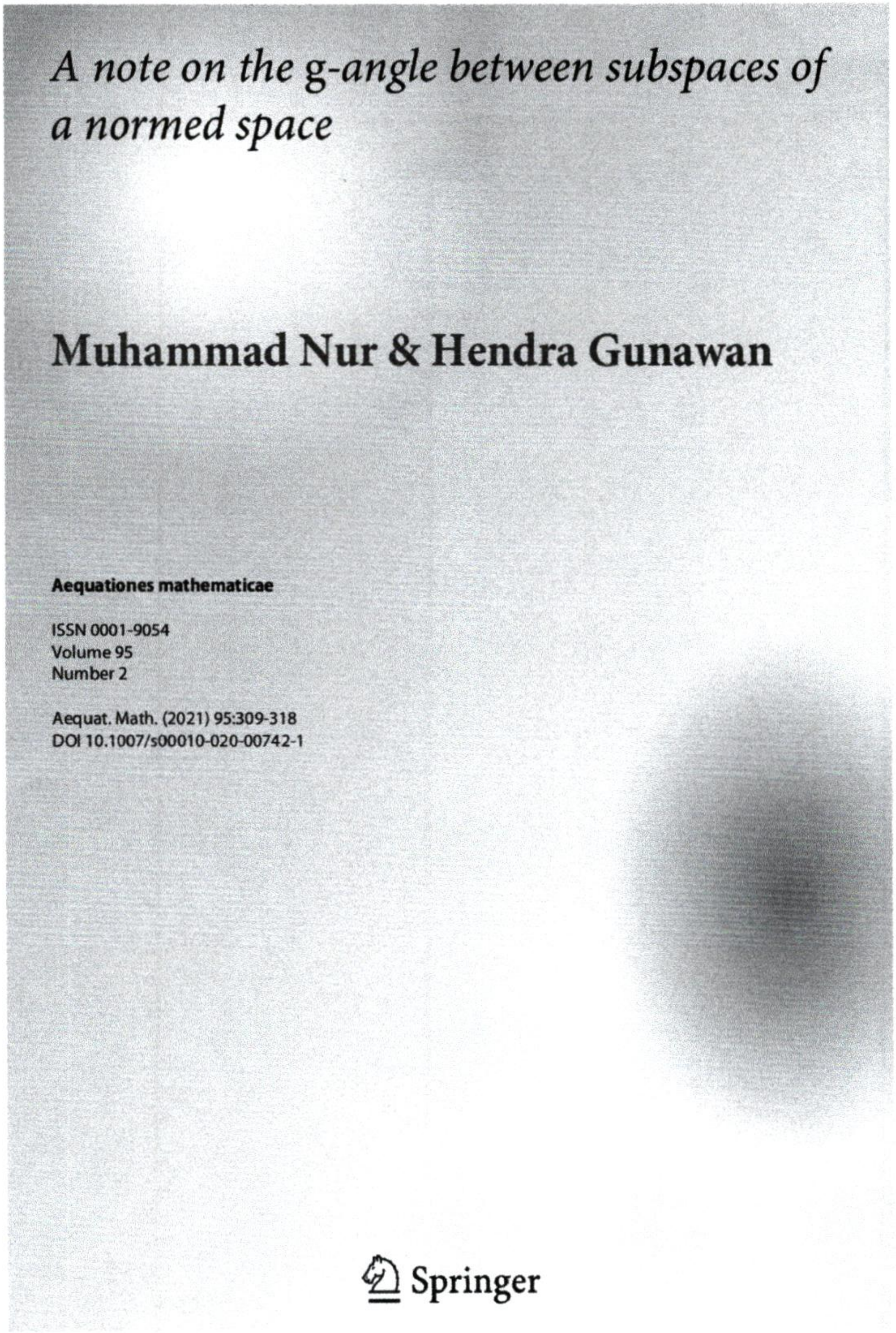

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A note on the g-angle between subspaces of a normed space

MUHAMMAD NUR AND HENDRA GUNAWAN

Abstract. We introduce a new 2-norm on a normed space using a semi-inner product g on the space. Using this 2-norm, we propose a formula for the g-angle between 2-dimensional subspaces in the space. Our formula serves as a revision of the one proposed by Nur et al. (Beitr. Algebra Geom. 59(1):133–143, 2018).

Mathematics Subject Classification. 15A03, 46B20, 51N15.

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1. Introduction

In an inner product space $(X, \langle \cdot, \cdot \rangle)$, we can calculate the angle $A(x, y)$ between two nonzero vectors x and y in X via the formula $A(x, y) := \frac{\langle x,y \rangle}{\|x\| \|y\|}$, where $\|x\| := \langle x, x \rangle^{\frac{1}{2}}$ denotes the induced norm in X. In 2005, Gunawan et al. [8] presented a formula for the angle between an n-dimensional subspace and an m-dimensional subspace of X (with $m \geq n$) by using the so-called standard n-norm on X.

Here we shall formulate the angle between 2-dimensional subspaces of a normed space, using a semi-inner product on the space. Let $(X, \| \cdot \|)$ be a normed space. The functional $g : X^2 \to \mathbb{R}$ defined by the formula

$$g(x, y) := \frac{1}{2} \|x\| \left[\tau_{+}(x, y) + \tau_{-}(x, y) \right],$$

with

$$\tau_{\pm}(x, y) := \lim_{t \to \pm 0} \frac{\|x + ty\| - \|x\|}{t},$$

clearly satisfies the following properties:

(1) $g(x, x) = \|x\|^2$ for every $x \in X$;
(2) $g(ax, by) = ab \cdot g(x, y)$ for every $x, y \in X$ and $a, b \in \mathbb{R}$;

(3) $g(x, x+y) = \|x\|^2 + g(x,y)$ for every $x, y \in X$;
(4) $|g(x,y)| \leq \|x\| \cdot \|y\|$ for every $x, y \in X$.

If, in addition, the functional $g(x,y)$ is linear in y, then g is called a *semi-inner product* on X. For example, the functional

$$g(x,y) := \|x\|_p^{2-p} \sum |\xi_k|^{p-1} \mathrm{sgn}\,(\xi_k)\,\eta_k, \quad x := (\xi_k)\,,\ y := (\eta_k) \in l^p,$$

is a semi-inner product on ℓ^p $(1 \leq p < \infty)$ [5,10]. Note that on an inner product space, the functional $g(x,y)$ is identical with the inner product $\langle \cdot, \cdot \rangle$.

Using a semi-inner product g on X, many researchers have studied the g-angle between two vectors, see, for example [1,10,13,14]. Recently, Nur et al. [15] investigated the g-angle between two subspaces of X. If $U = \mathrm{span}\{u\}$ is a 1-dimensional subspace and $V = \mathrm{span}\{v_1, \dots, v_m\}$ is an m-dimensional subspace of X with $m \geq 1$, then the g-angle between U and V is defined by $A_g(U,V)$ with $\cos^2 A_g(U,V) = \frac{\|u_V\|^2}{\|u\|^2}$. In this formula, u_V denotes the g-orthogonal projection of u on V. Likewise, if $U = \mathrm{span}\{u_1, u_2\}$ is a 2-dimensional subspace and $V = \mathrm{span}\{v_1, \dots, v_m\}$ is an m-dimensional subspace of X with $m \geq 2$, then the g-angle between U and V is defined by $\cos^2 A_g(U,V) = \frac{(\Lambda_p(u_{1V}, u_{2V}))^2}{(\Lambda_p(u_1, u_2))^2}$, where u_{iV}'s denote the g-orthogonal projections of u_i's on V, with $i = 1, 2$. This formula, however, depends on the choice of the basis for U, which is undesirable.

In this article, we will define a new 2-norm on X using a semi-inner product g. Recall that a 2-*norm* on a real vector space X is a mapping $\|\cdot,\cdot\| : X \times X \longrightarrow \mathbb{R}$ which satisfies the following four conditions:

(1) $\|x, y\| = 0$ if and only if x, y are linearly dependent;
(2) $\|x, y\|$ is invariant under permutation;
(3) $\|\alpha x, y\| = |\alpha| \|x, y\|$ for every $x, y \in X$ and for every $\alpha \in \mathbb{R}$;
(4) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$ for every $x, y, z \in X$.

The pair $(X, \|\cdot,\cdot\|)$ is called a 2-*normed space*. Geometrically, $\|x, y\|$ may be interpreted as the area of the 2-dimensional parallelepiped spanned by x and y. The theory of 2-normed spaces was first developed by Gähler [3]. Recent results can be found, for example, in [2,6,7].

Using a 2-norm, we will formulate the g-angle between two 2-dimensional subspaces of X, which serves as a revision of the formula derived in [15].

2. Main results

2.1. A new 2-norm

In this section, we will present a new 2-norm on a real normed space $(X, \|\cdot\|)$. (Unless otherwise stated, we shall always assume that X is a normed space.)

Let $g(\cdot,\cdot)$ be a semi-inner product on X. We define the mapping $\|\cdot,\cdot\|_g$ on X by

$$\|x_1, x_2\|_g = \sup_{y_j \in X, \|y_j\| \leq 1} \begin{vmatrix} g(y_1, x_1) & g(y_2, x_1) \\ g(y_1, x_2) & g(y_2, x_2) \end{vmatrix}. \tag{1}$$

The following fact tells us that the introduction of $\|\cdot,\cdot\|_g$ makes sense.

Fact 2.1. The inequality

$$\|x_1, x_2\|_g \leq 2\|x_1\|\|x_2\|$$

holds for every $x_1, x_2 \in X$.

Proof. Let $y_1, y_2 \in X$ with $\|y_1\| \leq 1$ and $\|y_2\| \leq 1$. By the triangle inequality for real numbers, we have

$$g(y_1, x_1)g(y_2, x_2) - g(y_2, x_1)g(y_1, x_2) \leq 2\|x_1\|\|x_2\|.$$

Hence, $\|x_1, x_2\|_g \leq 2\|x_1\|\|x_2\|$. ☐

Moreover, we have the following result.

Proposition 2.2. *The mapping* (1) *defines a 2-norm on* X.

Proof. We need to check that $\|\cdot,\cdot\|_g$ satisfies the four properties of a 2-norm.

1. Let $x_1 = kx_2$ with $k \in \mathbb{R}$. Observe that

   $$\begin{vmatrix} g(y_1, \text{[illegible]}) & g(y_2, x_1) \\ g(y_1, x_2) & g(y_2, x_2) \end{vmatrix} = 0.$$

   Hence, $\|x_1, x_2\|_g = 0$. Conversely, if $\|x_1, x_2\|_g = 0$, then the rows of the matrix

   $$\begin{bmatrix} g(y_1, x_1) & g(y_2, x_1) \\ g(y_1, x_2) & g(y_2, x_2) \end{bmatrix}$$

   [illegible] linearly dependent for all y_1, y_2 with $\|y_1\|, \|y_2\| \leq 1$. This happens only if x_1 and x_2 are linearly dependent.
2. By using properties of determinants, we obtain $\|x_1, x_2\|_g = \|x_2, x_1\|_g$.
3. Let $\alpha \in \mathbb{R}$. Using properties of determinants and the supremum, we obtain $\|\alpha x_1, x_2\|_g = |\alpha|\,\|x_1, x_2\|_g$.
4. Observe that for arbitrary $x_1, x_2, z \in X$, we obtain

$$\begin{aligned} &\|x_1, x_2 + z\|_g \\ &= \sup_{y_j \in X, \|y_j\| \leq 1} \begin{vmatrix} g(y_1, x_1) & g(y_2, x_1) \\ g(y_1, x\,\text{[illegible]}\, z) & g(y_2, x_2 + z) \end{vmatrix} \\ &= \sup_{y_j \in X, \|y_j\| \leq 1} \left\{ \begin{vmatrix} g(y_1, x_1) & g(y_2, x_1) \\ g(y_1, x_2) & g(y_2, x_2) \end{vmatrix} + \begin{vmatrix} (g(y_1, x_1) & g(y_2, x_1) \\ g(y_1, z) & g(y_2, z) \end{vmatrix} \right\} \\ &\leq \sup_{y_j \in X, \|y_j\| \leq 1} \begin{vmatrix} g(y_1, x_1) & g(y_2, x_1) \\ g(y_1, x_2) & g(y_2, x_2) \end{vmatrix} + \sup_{y_j \in X, \|y_j\| \leq 1} \begin{vmatrix} g(y_1, x_1) & g(y_2, x_1) \\ g(y_1, z) & g(y_2, z) \end{vmatrix} \\ &= \|x_1, x_2\|_g + \|x_1, z\|_g. \end{aligned}$$

This completes the proof. ☐

For an inner product space, we have the following fact.

Fact 2.3. Let $(X, \langle\cdot,\cdot\rangle)$ be a real inner product space. The two formulas for $\|\cdot,\cdot\|_g$ in (1) and for $\|\cdot,\cdot\|_s$ with

$$\|x_1, x_2\|_s := \begin{vmatrix} \langle x_1, x_1\rangle & \langle x_1, x_2\rangle \\ \langle x_2, x_1\rangle & \langle x_2, x_2\rangle \end{vmatrix}^{\frac{1}{2}}$$

for every $x_1, x_2 \in X$ are identical.

Proof. On the inner product space X, the semi-inner product $g(\cdot,\cdot)$ is identical with the inner product $\langle\cdot,\cdot\rangle$. Therefore,

$$\|x_1, x_2\|_g = \sup_{y_j \in X, \|y_j\| \le 1} \begin{vmatrix} \langle y_1, x_1\rangle & \langle y_1, x_2\rangle \\ \langle y_2, x_1\rangle & \langle y_2, x_2\rangle \end{vmatrix}.$$

By applying the generalized Cauchy–Schwarz inequality [11] and Hadamard's inequality [4], we obtain

$$\|x_1, x_2\|_g \le \sup_{y_j \in X, \|y_j\| \le 1} \|x_1, x_2\|_s \|y_1, y_2\|_s \le \|x_1, x_2\|_s.$$

Conversely, we assume now that $\{x_1, x_2\}$ are linearly independent. By using the Gram–Schmidt process, we have that $\{x_1', x_2'\}$ are orthogonal. Moreover, $\|x_1, x_2\|_s = \|x_1', x_2'\|_s = \|x_1'\|\|x_2'\|$. If $y_1 = \frac{x_1'}{\|x_1'\|}$ and $y_2 = \frac{x_2'}{\|x_2'\|}$, then $\|y_1\| = 1$ and $\|y_2\| = 1$. Next, using the properties of the inner product and determinants, we obtain

$$\begin{vmatrix} \langle y_1, x_1\rangle & \langle y_1, x_2\rangle \\ \langle y_2, x_1\rangle & \langle y_2, x_2\rangle \end{vmatrix} = \begin{vmatrix} \langle y_1, x_1'\rangle & \langle y_1, x_2'\rangle \\ \langle y_2, x_1'\rangle & \langle y_2, x_2'\rangle \end{vmatrix} = \frac{1}{\|x_1'\|\,\|x_2'\|} \begin{vmatrix} \langle x_1', x_1'\rangle & \langle x_1', x_2'\rangle \\ \langle x_2', x_1'\rangle & \langle x_2', x_2'\rangle \end{vmatrix}$$
$$= \|x_1'\|\,\|x_2'\| = \|x_1, x_2\|_s.$$

Thus, $\|x_1, x_2\|_g \ge \|x_1, x_2\|_s$, so that $\|x_1, x_2\|_g = \|x_1, x_2\|_s$. Next, if $\{x_1, x_2\}$ are linearly dependent, then $\|x_1, x_2\|_g = \|x_1, x_2\|_s = 0$. ☐

Note that in an inner product space we have a better inequality for Fact 2.1, namely $\|x_1, x_2\|_g \le \|x_1\|\|x_2\|$. This is Hadamard's inequality for $n = 2$ [4].

2.2. The g-angle between 2-dimensional subspaces

Here, using the 2-norm $\|\cdot,\cdot\|_g$, we will formulate the g-angle between 2-dimensional subspaces of X. First, we recall the definition of the g-orthogonal projection of u on a subspace of X as follows.

Definition 2.4. [12] Let u be a vector of X and $S = \text{span}\{x_1, \ldots, x_n\}$ be a subspace of X, with $\Gamma(x_1, \ldots, x_n) = \det[g(x_i, x_k)] \neq 0$. (This additional condition is added because it does not always follow from the condition of the linear independence.) The *g-orthogonal projection* of u on S, denoted by u_S, is defined by

$$u_S := -\frac{1}{\Gamma(x_1, \ldots, x_n)} \begin{vmatrix} 0 & x_1 & \cdots & x_n \\ g(x_1, u) & g(x_1, x_1) & \cdots & g(x_1, x_n) \\ \vdots & \vdots & \ddots & \vdots \\ g(x_n, u) & g(x_n, x_1) & \cdots & g(x_n, x_n) \end{vmatrix},$$

and its *g-orthogonal complement* $u - u_S$ is given by

$$u - u_S = \frac{1}{\Gamma(x_1, \ldots, x_n)} \begin{vmatrix} u & x_1 & \cdots & x_n \\ g(x_1, u) & g(x_1, x_1) & \cdots & g(x_1, x_n) \\ \vdots & \vdots & \ddots & \vdots \\ g(x_n, u) & g(x_n, x_1) & \cdots & g(x_n, x_n) \end{vmatrix}.$$

Note that the notation of the determinant $|\cdot|$ here has a special meaning because the elements of the matrix are not in the same field.

Let U and V be subspaces of X. Take arbitrary $u_1, u_2 \in U$ and $\alpha, \beta \in \mathbb{R}$. Using properties of determinants and the semi-inner product, we have $(\alpha u_1 + \beta u_2)_V = \alpha u_{1V} + \beta u_{2V}$. Hence, the g-orthogonal projection of U on V is a linear transformation from U to V.

Next, let $x_1, \ldots, x_n \in X$ be a finite sequence of linearly independent vectors. We can construct a *left g-orthonormal sequence* $x_1^*, \ldots, x_n^*$ with $x_1^* := \frac{x_1}{\|x_1\|}$ and

$$x_k^* := \frac{x_k - (x_k)_{S_{k-1}}}{\|x_k - (x_k)_{S_{k-1}}\|}, \tag{2}$$

where $S_{k-1} = \text{span}\left\{x_1^*, \ldots, x_{k-1}^*\right\}$. Note that $\text{span}\{x_1^*, \ldots, x_{k-1}^*\} = \text{span}\{x_1, \ldots, x_{k-1}\}$ for each $k = 2, \ldots, n$, and that $x_k^* \perp_g x_l^*$ for $k, l = 1, \ldots, n$ with $k < l$ (see [9, 12]). We also observe that $\Gamma(x_1^*, \ldots, x_{k-1}^*) = 1$ for each $k = 2, \ldots, n$.

From the properties of the 2-norm and the g-orthogonal projection, we have the following lemma.

Lemma 2.5. *If $U = \text{span}\{u_1, u_2\}$ and $V = \text{span}\{v_1, v_2\}$ are 2-dimensional subspaces of X, where $\{v_1, v_2\}$ is left g-orthonormal, then*

$$\begin{vmatrix} g(y_1, u_{1V}) & g(y_2, u_{1V}) \\ g(y_1, u_{2V}) & g(y_2, u_{2V}) \end{vmatrix} = \begin{vmatrix} g(v_1, u_1) & g(v_1, u_2) \\ g(v_2, u_1) & g(v_2, u_2) \end{vmatrix} \begin{vmatrix} g(y_1, v_1) & g(y_2, v_1) \\ g(y_1, v_2) & g(y_2, v_2) \end{vmatrix}$$

for every $y_1, y_2 \in X$.

Proof. If $\{v_1, v_2\}$ is left g-orthonormal, then $\Gamma\{v_1, v_2\} = 1$. Consequently,

$$g(y_j, u_{iV}) = -\begin{vmatrix} 0 & g(y_j, v_1) & g(y_j, v_2) \\ g(v_1, u_i) & 1 & 0 \\ g(v_2, u_i) & g(v_2, v_1) & 1 \end{vmatrix}$$

for $i, j = 1, 2$. By using the properties of determinants, we obtain

$$\begin{aligned}
&\begin{vmatrix} g(y_1, u_{1V}) & g(y_2, u_{1V}) \\ g(y_1, u_{2V}) & g(y_2, u_{2V}) \end{vmatrix} \\
&= \begin{vmatrix} 0 & g(y_1, v_1) & g(y_1, v_2) \\ g(v_1, u_1) & 1 & 0 \\ g(v_2, u_1) & g(v_2, v_1) & 1 \end{vmatrix} \begin{vmatrix} 0 & g(y_2, v_1) & g(y_2, v_2) \\ g(v_1, u_2) & 1 & 0 \\ g(v_2, u_2) & g(v_2, v_1) & 1 \end{vmatrix} \\
&\quad - \begin{vmatrix} 0 & g(y_2, v_1) & g(y_2, v_2) \\ g(v_1, u_1) & 1 & 0 \\ g(v_2, u_1) & g(v_2, v_1) & 1 \end{vmatrix} \begin{vmatrix} 0 & g(y_1, v_1) & g(y_1, v_2) \\ g(v_1, u_2) & 1 & 0 \\ g(v_2, u_2) & g(v_2, v_1) & 1 \end{vmatrix} \\
&= (-g(y_1, v_1)g(v_1, u_1) + g(y_1, v_2)g(v_1, u_1)g(v_2, v_1) - g(y_1, v_2)g(v_2, u_1)) \\
&\quad (-g(y_2, v_1)g(v_1, u_2) + g(y_2, v_2)g(v_1, u_2)g(v_2, v_1) - g(y_2, v_2)g(v_2, u_2)) \\
&\quad - (-g(y_2, v_1)g(v_1, u_1) + g(y_2, v_2)g(v_1, u_1)g(v_2, v_1) - g(y_2, v_2)g(v_2, u_1)) \\
&\quad (-g(y_1, v_1)g(v_1, u_2) + g(y_1, v_2)g(v_1, u_2)g(v_2, v_1) - g(y_1, v_2)g(v_2, u_2)) \\
&= g(y_1, v_1)g(y_2, v_2)g(v_1, u_1)g(v_2, u_2 + g(y_1, v_2)g(y_2, v_1)g(v_1, u_2)g(v_2, u_1) \\
&\quad - g(y_1, v_2)g(y_2, v_1)g(v_1, u_1)g(v_2, u_2) - g(y_1, v_1)g(y_2, v_2)g(v_2, u_1)g(v_1, u_2) \\
&= g(v_1, u_1)g(v_2, u_2)\begin{vmatrix} g(y_1, v_1) & g(y_2, v_1) \\ g(y_1, v_2) & g(y_2, v_2) \end{vmatrix} - g(v_1, u_2)g(v_2, u_1)\begin{vmatrix} g(y_1, v_1) & g(y_2, v_1) \\ g(y_1, v_2) & g(y_2, v_2) \end{vmatrix}.
\end{aligned}$$

Hence,

$$\begin{vmatrix} g(y_1, u_{1V}) & g(y_2, u_{1V}) \\ g(y_1, u_{2V}) & g(y_2, u_{2V}) \end{vmatrix} = \begin{vmatrix} g(v_1, u_1) & g(v_1, u_2) \\ g(v_2, u_1) & g(v_2, u_2) \end{vmatrix} \begin{vmatrix} g(y_1, v_1) & g(y_2, v_1) \\ g(y_1, v_2) & g(y_2, v_2) \end{vmatrix}.$$

This proves the lemma. ☐

Let us now define the g-angle between the 2-dimensional subspace $U = \mathrm{span}\{u_1, u_2\}$ and the 2-dimensional subspace V of X by

$$\cos^2 A_g(U, V) := \frac{\|u_{1V}, u_{2V}\|_g^2}{\|u_1, u_2\|_g^2 \sup\limits_{\mathrm{span}\{v_1, v_2\} = V} \|v_1^*, v_2^*\|_g^2}, \tag{3}$$

where u_{iV}'s denote the g-orthogonal projections of u_i's on V with $i = 1, 2$, and $\{v_1^*, v_2^*\}$ being the left g-orthonormal set obtained from $\{v_1, v_2\}$.

Remark 2.6. On an inner product space, the definition of the g-angle in (3) is identical with the angle defined in [8], namely

$$\cos^2 A_g(U, V) = \frac{\|u_{1V}, u_{2V}\|_g^2}{\|u_1, u_2\|_g^2}.$$

Remark 2.7. Let $\{v_1, v_2\}$ be a linearly independent set that spans V. If $v_1' = v_1$ and $v_2' = v_2 - \frac{g(v_1,v_2)}{\|v_1\|} v_1$, then $\left\{v_1', v_2'\right\}$ is left g-orthogonal. Likewise, if $w_1' = v_2$ and $w_2' = v_1 - \frac{g(v_2,v_1)}{\|v_2\|} v_2$, then $\{$[illegible]$, w_2'\}$ is also left g-orthogonal. Using properties of the 2-norm, we obtain $\|v_1, v_2\|_g = \|v_1', v_2'\|_g = \|w_1', w_2'\|_g$. But, in general, $\|v_1^*, v_2^*\|_g \neq \|w_1^*, w_2^*\|_g$, where $v_i^* = \frac{v_i'}{\|v_i'\|}$ and $w_i^* = \frac{w_i'}{\|w_i'\|}$ with $i = 1, 2$. For instance, take $v_1 = (1, 0, 0, \dots)$ and $v_2 = (1, 1, 0, \dots)$ in l^1 with the usual semi-inner product g. If $v_1' = v_1$ and $v_2' = v_2 - \frac{g(v_1,v_2)}{\|v_1\|} v_1 = (0, 1, 0, \dots)$, then $\|$[illegible] $\|v_2'\| = 1$. Next, if $w_1' = v_2$ and $w_2' = v_1 - \frac{g(v_2,v_1)}{\|v_2\|} v_2 = (\frac{1}{2}, -\frac{1}{2}, 0, \dots)$, then $\|w_1'\| \|w_2'\| = 2$. Hence $\|v_1^*, v_2^*\|_g \neq \|w_1^*, w_2^*\|_g$. Here we only change the order of the basis for V. Consider if we change the basis with another. This explains why we have the supremum term in the formula.

According to the following theorem, the definition of the g-angle in (3) makes sense.

Theorem 2.8. *The ratio on the right hand side of* (3) *is a number in* $[0, 1]$ *and is independent of the choice of a basis for* U *and* V .

Proof. Let $\{v_1, v_2\}$ be a linearly independent set that spans V. Using the process in (2), we obtain the left g-orthonormal set $\{v_1^*, v_2^*\}$. Notice that $\operatorname{span}\{v_1, v_2\} = \operatorname{span}\{v_1^*, v_2^*\}$. Using Lemma 2.5 and the definition of $\|\cdot,\cdot\|_g$ in (1), we have

$$\|u_{1V}, u_{2V}\|_g = \begin{vmatrix} g(v_1^*, u_1) & g(v_1^*, u_2) \\ g(v_2^*, u_1) & g(v_2^*, u_2) \end{vmatrix} \|v_1^*, v_2^*\|_g .$$

Since $\|v_i^*\| = 1$ for $i = 1, 2$, we have

$$\begin{aligned} \|u_{1V}, u_{2V}\|_g &= \begin{vmatrix} g(v_1^*, u_1) & g(v_1^*, u_2) \\ g(v_2^*, u_1) & g(v_2^*, u_2) \end{vmatrix} \|v_1^*, v_2^*\|_g \\ &\leq \sup_{y_j \in X, \|y_j\| \leq 1} \begin{vmatrix} g(y_1, u_1) & g(y_2, u_1) \\ g(y_1, u_2) & g(y_2, u_2) \end{vmatrix} \|v_1^*, v_2^*\|_g \\ &= \|u_1, u_2\|_g \|v_1^*, v_2^*\|_g \\ &\leq \|u_1, u_2\|_g \sup_{\operatorname{span}\{w_1, w_2\} = V} \|w_1^*, w_2^*\|_g , \end{aligned}$$

so that

$$\frac{\|u_{1V}, u_{2V}\|_g}{\|u_1, u_2\|_g \sup\limits_{\operatorname{span}\{w_1, w_2\} = V} \|w_1^*, w_2^*\|_g} \leq 1.$$

Secondly, note that the g-orthogonal projections of u_i's on V are independent of the choice of the basis for V [12]. Moreover, since the g-orthogonal projection of U on V is a linear transformation from U to V, the ratio of (3) is also invariant under every change of the basis for U. Indeed, the ratio is unchanged if we swap u_1 and u_2, replace u_1 with $u_1 + \alpha u_2$, replace u_1 with αu_1 or u_2 with αu_2 where $\alpha \neq 0$. The proof is complete. ☐

3. Concluding remarks

The formula (3) can be used to compute the g-angle between two subspaces of ℓ^p as follows. Let $\{v_1, \dots, v_m\}$ be a linearly independent set that spans V in ℓ^p. Using the process in (2), we obtain the left g-orthonormal set $\{v_1^*, \dots, v_m^*\}$. Here span $\{v_1, \dots, v_m\} =$ span $\{v_1^*, \dots, v_m^*\}$. Hence, for $i = 1, 2$,

$$u_{iV} = -\begin{vmatrix} 0 & v_1^* & \cdots & v_m^* \\ g(v_1^*, u_i) & g(v_1^*, v_1^*) & \cdots & g(v_1^*, v_m^*) \\ \vdots & \vdots & \ddots & \vdots \\ g(v_m^*, u_i) & g(v_m^*, v_1^*) & \cdots & g(v_m^*, v_m^*) \end{vmatrix} = -\begin{vmatrix} g(v_1^*, v_1^*) & \cdots & g(v_m^*, v_1^*) & v_1^* \\ \vdots & \ddots & \vdots & \vdots \\ g(v_1^*, v_m^*) & \cdots & g(v_m^*, v_m^*) & v_m^* \\ g(v_1^*, u_i) & \cdots & g(v_m^*, u_i) & 0 \end{vmatrix}.$$

Substituting $g(v_k^*, v_i^*) = \|v_k^*\|_p^{2-p} \sum_{j_k} |v_{kj_k}^*|^{p-1} \mathrm{sgn}(v_{kj_k}^*) v_{ij_k}^*$, we obtain

$$u_{iV} = -\sum_{j_m} \cdots \sum_{j_1} |v_{1j_m}^*|^{p-1} sgn(v_{1j_m}^*)$$
$$\times \cdots |v_{mj_1}^*|^{p-1} sgn(v_{mj_1}^*) \begin{vmatrix} v_{1j_1}^* & \cdots & v_{1j_m}^* & v_1^* \\ \vdots & \ddots & \vdots & \vdots \\ v_{mj_1}^* & \cdots & v_{mj_m}^* & v_m^* \\ u_{ij_1} & \cdots & u_{ij_m} & 0 \end{vmatrix}.$$

Using this formula, we can compute the value of the g-angle between two subspaces $U = \mathrm{span}\{u_1, u_2\}$ and $V = \mathrm{span}\{v_1, v_2\}$ of ℓ^p for $1 \leq p < \infty$. For instance, in ℓ^2, let $U = \mathrm{span}\{u_1, u_2\}$ and $V = \mathrm{span}\{v_1, v_2\}$ with $u_1 = (1, 1, 2, 0, \dots)$, $u_2 = (2, 1, 3, 0, \dots)$, $v_1 = (1, 0, 0, 0, \dots)$, and $v_2 = (0, 1, 0, 0, \dots)$. We obtain $u_{1V} = (1, 1, 0, 0, \dots)$ and $u_{2V} = (2, 1, 0, 0, 0, \dots)$. Moreover, $\|u_1\| = \sqrt{6}$, $\|u_2\| = \sqrt{14}$, $\|u_{1V}\| = \sqrt{2}$, and $\|u_{2V}\| = \sqrt{5}$. Observe that $\sup\limits_{\mathrm{span}\{w_1, w_2\} = V} \|w_1^*, w_2^*\|_g = \|w_1^*, w_2^*\|_s = 1$. Next

$$\|u_{1V}, u_{2V}\|_g = \|u_{1V}, u_{2V}\|_s = \sqrt{10 - 9} = 1$$

and

$$\|u_1, u_2\|_g = \|u_1, u_2\|_s = \sqrt{84 - 81} = \sqrt{3}.$$

Thus $\cos^2 A_g(U, V) = \frac{1}{3}$, so that $A_g(U, V) = \arccos(\frac{1}{3}\sqrt{3})$.

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