



# Three Equivalent $n$ -Norms on the Space of $p$ -Summable Sequences

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## Abstract

Given a normed space, one can define a new  $n$ -norm using a semi-inner product  $g$  on the space, different from the  $n$ -norm defined by Gähler. In this paper, we are interested in the new  $n$ -norm which is defined using such a functional  $g$  on the space  $\ell^p$  of  $p$ -summable sequences, where  $1 \leq p < \infty$ . We prove particularly that the new  $n$ -norm is equivalent with the one defined previously by Gunawan on  $\ell^p$ .

## 1. Introduction

On a normed space  $(X, \|\cdot\|)$ , let  $g : X^2 \rightarrow \mathbb{R}$  be the functional defined by the formula

$$g(x, y) := \frac{1}{2} \|x\| [\tau_+(x, y) + \tau_-(x, y)],$$

with

$$\tau_{\pm}(x, y) := \lim_{t \rightarrow 0^{\pm}} \frac{\|x + ty\| - \|x\|}{t}.$$

Then, one may check that  $g$  satisfies the following properties:

- (1)  $g(x, x) = \|x\|^2$  for every  $x \in X$ ;
- (2)  $g(\alpha x, \beta y) = \alpha\beta g(x, y)$  for every  $x, y \in X$  and  $\alpha, \beta \in \mathbb{R}$ ;
- (3)  $g(x, x+y) = \|x\|^2 + g(x, y)$  for every  $x, y \in X$ ;
- (4)  $|g(x, y)| \leq \|x\| \|y\|$  for every  $x, y \in X$ .

Assuming that the  $g$ -functional is linear in the second argument then  $[y, x] = g(x, y)$  is a *semi-inner product* on  $X$ .

Note that all vector spaces in text are assumed to be over  $\mathbb{R}$ . For example, one may observe that the functional

$$g(x, y) := \|x\|_p^{2-p} \sum_k |x_k|^{p-1} \operatorname{sgn}(x_k) y_k, \quad x := (x_k), y := (y_k) \in \ell^p$$

is a semi-inner product on  $\ell^p$ ,  $1 \leq p < \infty$  [1].

**Remark 1.1.** Note that not all vector spaces have the property that the  $g$ -functional is linear in the second argument. If the normed space is smooth, then the  $g$ -functional is linear in the second argument. A normed spaces with the property that the  $g$ -functional is linear in the second argument is referred to as normed spaces of (G)-type [2].

By using a semi-inner product  $g$ , Miličić [3] introduced the following orthogonality relation on  $X$ :  $x$  is said to be  $g$ -orthogonal to  $y$ , denoted by  $x \perp_g y$ , provided that  $g(x, y) = 0$ . For more recent works, see in [4, 5].

Recently, Nur and Gunawan in [6] defined a 2-norm on  $X$  by

$$\|x_1, x_2\|_g := \sup_{\|y_j\| \leq 1, j=1,2} \begin{vmatrix} g(y_1, x_1) & g(y_2, x_1) \\ g(y_1, x_2) & g(y_2, x_2) \end{vmatrix}.$$

Similarly, we can define an  $n$ -norm (with  $n \geq 2$ ) using the semi-inner product  $g$  on  $X$ . An  $n$ -norm on  $X$  is a mapping  $\|\cdot, \dots, \cdot\| : X \times \dots \times X \rightarrow \mathbb{R}$  which satisfies the following four properties:

- (1)  $\|x_1, \dots, x_n\| = 0$  if and only if  $x_1, \dots, x_n$  are linearly dependent;
- (2)  $\|x_1, \dots, x_n\|$  is invariant under permutation;
- (3)  $\|\alpha x_1, \dots, x_n\| = |\alpha| \|x_1, \dots, x_n\|$  for every  $x_1, \dots, x_n \in X$  and for every  $\alpha \in \mathbb{R}$ ;
- (4)  $\|x_1, \dots, x_{n-1}, y + z\| \leq \|x_1, \dots, x_{n-1}, y\| + \|x_1, \dots, x_{n-1}, z\|$  for every  $x, y, z \in X$ .

The pair  $(X, \|\cdot, \dots, \cdot\|)$  is called an  $n$ -normed space.

The theory of 2-normed spaces was initially introduced by Gähler [7] in the 1960's. Meanwhile, the theory of  $n$ -normed spaces for  $n \geq 2$  was developed in [8]-[10]. See [11]-[15] for recent results on this subject.

On the space  $\ell^p$  of  $p$ -summable sequences, where  $1 \leq p < \infty$ , the following  $n$ -norm

$$\|x_1, \dots, x_n\|_p := \left[ \frac{1}{n!} \sum_{k_1} \dots \sum_{k_n} \left( \text{abs} \begin{vmatrix} x_{1k_1} & \dots & x_{1k_n} \\ \vdots & \ddots & \vdots \\ x_{nk_1} & \dots & x_{nk_n} \end{vmatrix} \right)^p \right]^{\frac{1}{p}} \tag{1.1}$$

is defined by Gunawan in [16]. As shown in [17, 18], this  $n$ -norm is equivalent with the one formulated by Gähler in [8]-[10], namely

$$\|x_1, \dots, x_n\|_p' := \sup_{\|y_j\|_p' \leq 1, j=1, \dots, n} \begin{vmatrix} \sum_k x_{1k} y_{1k} & \dots & \sum_k x_{1k} y_{nk} \\ \vdots & \ddots & \vdots \\ \sum_k x_{nk} y_{1k} & \dots & \sum_k x_{nk} y_{nk} \end{vmatrix}, \tag{1.2}$$

where  $p'$  denotes the dual exponent of  $p$ . Precisely, we have the following theorem.

**Theorem 1.2.** [19] For every  $x_1, \dots, x_n \in \ell^p$  ( $1 \leq p < \infty$ ), we have

$$(n!)^{\frac{1}{p}-1} \|x_1, \dots, x_n\|_p \leq \|x_1, \dots, x_n\|_p' \leq (n!)^{\frac{1}{p}} \|x_1, \dots, x_n\|_p.$$

In this article, we shall first prove that, on  $\ell^p$  ( $1 \leq p < \infty$ ), the new 2-norm  $\|\cdot, \cdot\|_g$  is equivalent with the 2-norm  $\|\cdot, \cdot\|_p$  which is defined in (1.1). Using this result, we can prove that the 2-normed space  $(\ell^p, \|\cdot, \cdot\|_g)$  is complete. We then extend the result for all  $n \geq 2$ .

## 2. Main results

Before we discuss the equivalence between the two 2-norms on  $\ell^p$  ( $1 \leq p < \infty$ ), we need some definitions. Let  $(X, \|\cdot\|)$  be a normed space. We define the  $g$ -orthogonal projection of a vector  $y$  on a subspace  $S$  of  $X$  as follows.

**Definition 2.1.** [20] Let  $y \in X$  and  $S = \text{span}\{x_1, \dots, x_m\}$  be a subspace of  $X$  with  $\Gamma(x_1, \dots, x_m) = \det[g(x_i, x_j)] \neq 0$ . The  $g$ -orthogonal projection of  $y$  on  $S$ , which we denote by  $y_S$ , is defined by

$$y_S := \frac{1}{\Gamma(x_1, \dots, x_m)} \begin{vmatrix} 0 & x_1 & \dots & x_m \\ g(x_1, y) & g(x_1, x_1) & \dots & g(x_1, x_m) \\ \vdots & \vdots & \ddots & \vdots \\ g(x_m, y) & g(x_m, x_1) & \dots & g(x_m, x_m) \end{vmatrix},$$

and its  $g$ -orthogonal complement  $y - y_S$  is given by

$$y - y_S = \frac{1}{\Gamma(x_1, \dots, x_m)} \begin{vmatrix} y & x_1 & \dots & x_m \\ g(x_1, y) & g(x_1, x_1) & \dots & g(x_1, x_m) \\ \vdots & \vdots & \ddots & \vdots \\ g(x_m, y) & g(x_m, x_1) & \dots & g(x_m, x_m) \end{vmatrix}.$$

Observe here that  $x_i \perp_g y - y_S$  for every  $i = 1, \dots, m$ . Note that, if  $S = \text{span}\{x\}$ , then

$$y_S = \frac{g(x, y)}{\|x\|^2} x,$$

and  $y - y_S$  is the  $g$ -orthogonal complement  $y$  on  $S$ . It is clear here that  $x \perp_g y - y_S$ .

Next, let  $x_1, \dots, x_n \in X$  be a set of  $n$  linearly independent vectors. We may construct a *left g-orthogonal sequence*  $x_1^*, \dots, x_n^*$  with  $x_1^* := x_1$ , and

$$x_i^* := x_i - (x_i)_{S_{i-1}}, \tag{2.1}$$

where  $S_{i-1} = \text{span} \{x_1^*, \dots, x_{i-1}^*\}$  for  $i = 2, \dots, n$ . Observe here that  $x_i^* \perp_g x_j^*$  for  $i < j$  (see [15, 20]).

For  $X = \ell^p$  ( $1 \leq p < \infty$ ), we have relation for the  $n$ -norm  $\|x_1, \dots, x_n\|_p$  and the ‘volume’ of the  $n$ -dimensional parallelepiped spanned by  $\{x_1, \dots, x_n\}$  in  $\ell^p$ , namely  $V(x_1, \dots, x_n) = \prod_{i=1}^n \|x_i^*\|_p$ , as follows.

**Theorem 2.2.** [19] *Let  $\{x_1, \dots, x_n\}$  be a set of linearly independent vectors in  $\ell^p$  ( $1 \leq p < \infty$ ). Then we have*

$$(n!)^{\frac{1}{p}-1} \|x_1, \dots, x_n\|_p \leq V(x_{i_1}, \dots, x_{i_n}) \leq (n!)^{\frac{1}{p}} \|x_1, \dots, x_n\|_p$$

for any permutation  $(i_1, \dots, i_n)$  of  $(1, \dots, n)$ .

Note that the value of  $V(x_1, \dots, x_n)$  may not be invariant under permutation of  $(x_1, \dots, x_n)$  because  $g(\cdot, \cdot)$  may not be symmetry. The above theorem states that all possible values of  $V(x_{i_1}, \dots, x_{i_n})$  lie between two multiples of  $\|x_1, \dots, x_n\|_p$ , independent of the permutation.

### 2.1. The equivalence between two 2-norms

Let us consider Gunawan’s definition and Gähler’s definition of 2-norm on  $\ell^p$  ( $1 \leq p < \infty$ ), namely:

$$\|x_1, x_2\|_p = \left[ \sum_{k_1} \sum_{k_2} \left( \text{abs} \begin{vmatrix} x_{1k_1} & x_{1k_2} \\ x_{2k_1} & x_{2k_2} \end{vmatrix} \right)^p \right]^{\frac{1}{p}}$$

and

$$\|x_1, x_2\|'_p := \sup_{\|y_j\|_{p'} \leq 1, j=1,2} \left| \begin{vmatrix} \sum_k x_{1k} y_{1k} & \sum_k x_{1k} y_{2k} \\ \sum_k x_{2k} y_{1k} & \sum_k x_{2k} y_{2k} \end{vmatrix} \right|.$$

Meanwhile, Nur and Gunawan’s 2-norm is given by

$$\|x_1, x_2\|_{g,p} = \sup_{\|y_j\|_p \leq 1, j=1,2} \left| \begin{vmatrix} \|y_1\|_p^{2-p} \sum_k |y_{1k}|^{p-1} \text{sgn}(y_{1k}) x_{1k} & \|y_2\|_p^{2-p} \sum_k |y_{2k}|^{p-1} \text{sgn}(y_{2k}) x_{1k} \\ \|y_1\|_p^{2-p} \sum_k |y_{1k}|^{p-1} \text{sgn}(y_{1k}) x_{2k} & \|y_2\|_p^{2-p} \sum_k |y_{2k}|^{p-1} \text{sgn}(y_{2k}) x_{2k} \end{vmatrix} \right|.$$

**Remark 2.3.** *Using properties of determinants, the above 2-norm may be rewritten as*

$$\|x_1, x_2\|_{g,p} = \sup_{\|y_j\|_p \leq 1, j=1,2} \frac{1}{2} \prod_{j=1}^2 \|y_j\|_p^{2-p} \sum_{k_1} \sum_{k_2} \left| \begin{vmatrix} |y_{1k_1}|^{p-1} \text{sgn}(y_{1k_1}) & |y_{1k_2}|^{p-1} \text{sgn}(y_{1k_2}) \\ |y_{2k_1}|^{p-1} \text{sgn}(y_{2k_1}) & |y_{2k_2}|^{p-1} \text{sgn}(y_{2k_2}) \end{vmatrix} \right| \left\| \begin{vmatrix} x_{1k_1} & x_{1k_2} \\ x_{2k_1} & x_{2k_2} \end{vmatrix} \right|.$$

For  $p = 2$ , we observe that

$$\|x_1, x_2\|_{g,2} = \sup_{\|y_j\|_2 \leq 1, j=1,2} \frac{1}{2} \sum_{k_1} \sum_{k_2} \left\| \begin{vmatrix} y_{1k_1} & y_{1k_2} \\ y_{2k_1} & y_{2k_2} \end{vmatrix} \right\| \left\| \begin{vmatrix} x_{1k_1} & x_{1k_2} \\ x_{2k_1} & x_{2k_2} \end{vmatrix} \right\|.$$

One may then verify that the three 2-norms  $\|\cdot, \cdot\|_2$ ,  $\|\cdot, \cdot\|'_2$  and  $\|\cdot, \cdot\|_{g,2}$  are identical (see [6, 12]).

For other values of  $p$ , we have the following theorem.

**Theorem 2.4.** *For every  $x_1, x_2 \in \ell^p$  ( $1 \leq p < \infty$ ), we have*

$$2^{\frac{1}{p}-1} \|x_1, x_2\|_p \leq \|x_1, x_2\|_{g,p} \leq \|x_1, x_2\|'_p \leq 2^{\frac{1}{p}} \|x_1, x_2\|_p.$$

*Proof.* For  $j = 1, 2$ , let  $y_j \in \ell^p$  with  $\|y_j\|_p \leq 1$ . Take  $u_j = (u_{jk})$  with  $u_{jk} = \|y_j\|_p^{2-p} |y_{jk}|^{p-1} \text{sgn}(y_{jk})$ . We observe that  $u_j \in \ell^{p'}$  with  $\|u_j\|_{p'} = \|y_j\|_p$ . As a consequence, we have  $\|x_1, x_2\|_{g,p} \leq \|x_1, x_2\|'_p$ . By using Theorem 1.2, we obtain

$$\|x_1, x_2\|_{g,p} \leq \|x_1, x_2\|'_p \leq 2^{\frac{1}{p}} \|x_1, x_2\|_p.$$

Next, assume that  $\{x_1, x_2\}$  is linearly independent. Using the process in (2.1), we obtain the left  $g$ -orthogonal set  $\{x_1^*, x_2^*\}$ . Then, by Theorem 2.2, we have

$$2^{\frac{1}{p}-1} \|x_1, x_2\|_p \leq V(x_1, x_2) = \|x_1^*\|_p \|x_2^*\|_p.$$

For  $j = 1, 2$ , let  $y_j = \frac{x_j^*}{\|x_j^*\|_p}$ , so that  $\|y_j\|_p = 1$ . It follows from the properties of semi-inner product  $g$  and matrix determinants that

$$\begin{aligned} \begin{vmatrix} g(y_1, x_1) & g(y_2, x_1) \\ g(y_1, x_2) & g(y_2, x_2) \end{vmatrix} &= \begin{vmatrix} \frac{1}{\|x_1^*\|_p} g(x_1^*, x_1^*) & \frac{1}{\|x_2^*\|_p} g(x_2^*, x_1^*) \\ \frac{1}{\|x_1^*\|_p} g(x_1^*, x_2^*) & \frac{1}{\|x_2^*\|_p} g(x_2^*, x_2^*) \end{vmatrix} \\ &= \|x_1^*\|_p \|x_2^*\|_p = V(x_1, x_2) \\ &\geq 2^{\frac{1}{p}-1} \|x_1, x_2\|_p. \end{aligned}$$

By the definition of  $\|\cdot, \cdot\|_{g,p}$ , we conclude that  $\|x_1, x_2\|_{g,p} \geq 2^{\frac{1}{p}-1} \|x_1, x_2\|_p$ . Combining with the previous inequalities, we have

$$2^{\frac{1}{p}-1} \|x_1, x_2\|_p \leq \|x_1, x_2\|_{g,p} \leq \|x_1, x_2\|_p' \leq 2^{\frac{1}{p}} \|x_1, x_2\|_p.$$

Note that if  $\{x_1, x_2\}$  is a linearly dependent set, then all the 2-norms are equal 0, and so we have the equalities.  $\square$

**Corollary 2.5.** For  $1 \leq p < \infty$ , the three 2-norms  $\|\cdot, \cdot\|_{g,p}$ ,  $\|\cdot, \cdot\|_p'$ , and  $\|\cdot, \cdot\|_p$  are pairwise equivalent.

Since  $(\ell^p, \|\cdot, \cdot\|_p)$  is a 2-Banach space [1], we obtain the following corollary.

**Corollary 2.6.** For  $1 \leq p < \infty$ , the 2-normed space  $(\ell^p, \|\cdot, \cdot\|_{g,p})$  is a 2-Banach space.

## 2.2. The equivalence between two $n$ -norms

All results in above subsection can be extended to  $n$ -normed spaces for any  $n \geq 2$ . Suppose that  $g$  is a semi-inner product on  $(X, \|\cdot\|)$ . Consider the following mapping  $\|\cdot, \dots, \cdot\|_g$  on  $X \times \dots \times X$ :

$$\|x_1, \dots, x_n\|_g = \sup_{\|y_j\| \leq 1, j=1, \dots, n} \begin{vmatrix} g(y_1, x_1) & \dots & g(y_n, x_1) \\ \vdots & \ddots & \vdots \\ g(y_1, x_n) & \dots & g(y_n, x_n) \end{vmatrix} = \sup_{\|y_j\| \leq 1, j=1, \dots, n} \det[g(y_j, x_i)]. \quad (2.2)$$

If  $\|y_j\| \leq 1$  for  $j = 1, \dots, n$ , then  $\det[g(y_j, x_i)] \leq n! \prod_{i=1}^n \|x_i\|$ . Note that the factor  $n!$  comes from the number of terms in the expansion of  $\det[g(y_j, x_i)]$ . The following fact tells us that  $\|\cdot, \dots, \cdot\|_g$  is a finite number.

**Fact 2.7.** The inequality

$$\|x_1, \dots, x_n\|_g \leq n! \prod_{i=1}^n \|x_i\|$$

holds whenever  $x_1, \dots, x_n \in X$ .

Moreover, we have the following result.

**Proposition 2.8.** The mapping (2.2) defines an  $n$ -norm on  $X$ .

*Proof.* It is obvious that, if  $\{x_1, \dots, x_n\}$  is linearly dependent, then we have  $\|x_1, \dots, x_n\|_g = 0$ . Conversely, if  $\|x_1, \dots, x_n\|_g = 0$ , then the rows of the matrix  $[g(y_j, x_i)]$  are linearly dependent for every  $y_1, \dots, y_n \in X$  with  $\|y_j\| \leq 1$ ,  $j = 1, \dots, n$ . This happens only if  $x_1, \dots, x_n$  are linearly dependent.

Next, by using the properties of supremum and matrix determinants, we obtain the invariance of  $\|x_1, \dots, x_n\|_g$  under permutation. Furthermore, we have  $\|\alpha x_1, \dots, \alpha x_n\|_g = |\alpha| \|x_1, \dots, x_n\|_g$  for  $\alpha \in \mathbb{R}$ .

Finally, for arbitrary  $x_0, x_1, \dots, x_n \in X$ , we obtain

$$\begin{aligned} \|x_0 + x_1, \dots, x_n\|_g &= \sup_{\|y_j\| \leq 1, j=1, \dots, n} \begin{vmatrix} g(y_1, x_0 + x_1) & \dots & g(y_n, x_0 + x_1) \\ \vdots & \ddots & \vdots \\ g(y_1, x_n) & \dots & g(y_n, x_n) \end{vmatrix} \\ &\leq \sup_{\|y_j\| \leq 1, j=1, \dots, n} \begin{vmatrix} g(y_1, x_0) & \dots & g(y_n, x_0) \\ \vdots & \ddots & \vdots \\ g(y_1, x_n) & \dots & g(y_n, x_n) \end{vmatrix} + \sup_{\|y_j\| \leq 1, j=1, \dots, n} \begin{vmatrix} g(y_1, x_1) & \dots & g(y_n, x_1) \\ \vdots & \ddots & \vdots \\ g(y_1, x_n) & \dots & g(y_n, x_n) \end{vmatrix} \\ &= \|x_0, \dots, x_n\|_g + \|x_1, \dots, x_n\|_g. \end{aligned}$$

This completes the proof.  $\square$

The following theorem holds for an inner product space  $(X, \langle \cdot, \cdot \rangle)$ .

**Theorem 2.9.** If  $(X, \langle \cdot, \cdot \rangle)$  is a real inner product space, then the two  $n$ -norms  $\|\cdot, \dots, \cdot\|_g$  in (2.2) and  $\|\cdot, \dots, \cdot\|_s$  given by

$$\|x_1, \dots, x_n\|_s := \left| \begin{array}{ccc} \langle x_1, x_1 \rangle & \cdots & \langle x_n, x_1 \rangle \\ \vdots & \ddots & \vdots \\ \langle x_1, x_n \rangle & \cdots & \langle x_n, x_n \rangle \end{array} \right|^{\frac{1}{2}}$$

are identical.

*Proof.* On the inner product space  $X$ , the functional  $g(\cdot, \cdot)$  is identical with the inner product  $\langle \cdot, \cdot \rangle$ . Therefore,

$$\|x_1, \dots, x_n\|_g = \sup_{\|y_j\| \leq 1, j=1, \dots, n} \left| \begin{array}{ccc} \langle y_1, x_1 \rangle & \cdots & \langle y_n, x_1 \rangle \\ \vdots & \ddots & \vdots \\ \langle y_1, x_n \rangle & \cdots & \langle y_n, x_n \rangle \end{array} \right|.$$

Now, applying the generalized Cauchy-Schwarz inequality [21] and Hadamard's inequality [22], we get

$$\|x_1, \dots, x_n\|_g \leq \sup_{\|y_j\| \leq 1, j=1, \dots, n} \|x_1, \dots, x_n\|_s \|y_1, \dots, y_n\|_s \leq \|x_1, \dots, x_n\|_s.$$

Conversely, suppose that  $\{x_1, \dots, x_n\}$  is linearly independent. Using the Gram-Schmidt process, we get the orthogonal set  $\{x'_1, \dots, x'_n\}$ . Because the determinant of the Gram matrix of a linearly independent set being equal to the Gram matrix of the associated orthogonal set (obtained using Gram-Schmidt process), we have  $\|x_1, \dots, x_n\|_s = \|x'_1, \dots, x'_n\|_s = \|x'_1\| \cdots \|x'_n\|$ . For  $j = 1, \dots, n$ , let  $y_j = \frac{x'_j}{\|x'_j\|}$ , so that  $\|y_j\| = 1$ . Then, by the properties of the inner product and matrix determinants, we obtain

$$\begin{aligned} \left| \begin{array}{ccc} \langle y_1, x_1 \rangle & \cdots & \langle y_n, x_1 \rangle \\ \vdots & \ddots & \vdots \\ \langle y_1, x_n \rangle & \cdots & \langle y_n, x_n \rangle \end{array} \right| &= \left| \begin{array}{ccc} \langle y_1, x'_1 \rangle & \cdots & \langle y_n, x'_1 \rangle \\ \vdots & \ddots & \vdots \\ \langle y_1, x'_n \rangle & \cdots & \langle y_n, x'_n \rangle \end{array} \right| = \frac{1}{\|x'_1\| \cdots \|x'_n\|} \left| \begin{array}{ccc} \langle x'_1, x'_1 \rangle & \cdots & \langle x'_n, x'_1 \rangle \\ \vdots & \ddots & \vdots \\ \langle x'_1, x'_n \rangle & \cdots & \langle x'_n, x'_n \rangle \end{array} \right| \\ &= \|x'_1\| \cdots \|x'_n\| = \|x_1, \dots, x_n\|_s. \end{aligned}$$

Thus,  $\|x_1, \dots, x_n\|_g \geq \|x_1, \dots, x_n\|_s$ . Hence we conclude that  $\|x_1, \dots, x_n\|_g = \|x_1, \dots, x_n\|_s$  whenever  $\{x_1, \dots, x_n\}$  is linearly independent. If  $\{x_1, \dots, x_n\}$  is linearly dependent, then  $\|x_1, \dots, x_n\|_g = \|x_1, \dots, x_n\|_s = 0$ .  $\square$

**Remark 2.10.** Note that, in an inner product space, we have the well-known Hadamard's inequality [22]

$$\|x_1, \dots, x_n\|_g = \|x_1, \dots, x_n\|_s \leq \|x_1\| \cdots \|x_n\|,$$

which is better than that in Fact ??.

For  $X = \ell^p$  ( $1 \leq p < \infty$ ), we rewrite the formula in (2.2) as

$$\|x_1, \dots, x_n\|_{g,p} = \sup_{\|y_j\|_p \leq 1, j=1, \dots, n} \left| \begin{array}{ccc} g(y_1, x_1) & \cdots & g(y_n, x_1) \\ \vdots & \ddots & \vdots \\ g(y_1, x_n) & \cdots & g(y_n, x_n) \end{array} \right|.$$

Substituting  $g(y_j, x_i) = \|y_j\|_p^{2-p} \sum_k |y_{jk}|^{p-1} \text{sgn}(y_{jk}) x_{ik}$  and using the properties of determinants, we have

$$\begin{aligned} \|x_1, \dots, x_n\|_{g,p} &= \sup_{\|y_j\|_p \leq 1, j=1, \dots, n} \left| \begin{array}{ccc} \|y_1\|_p^{2-p} \sum_k |y_{1k}|^{p-1} \text{sgn}(y_{1k}) x_{1k} & \cdots & \|y_n\|_p^{2-p} \sum_k |y_{nk}|^{p-1} \text{sgn}(y_{nk}) x_{1k} \\ \vdots & \ddots & \vdots \\ \|y_1\|_p^{2-p} \sum_k |y_{1k}|^{p-1} \text{sgn}(y_{1k}) x_{nk} & \cdots & \|y_n\|_p^{2-p} \sum_k |y_{nk}|^{p-1} \text{sgn}(y_{nk}) x_{nk} \end{array} \right| \\ &= \sup_{\|y_j\|_p \leq 1, j=1, \dots, n} \prod_{j=1}^n \|y_j\|_p^{2-p} \sum_{k_1} \cdots \sum_{k_n} \prod_{j=1}^n |y_{jk_j}|^{p-1} \text{sgn}(y_{jk_j}) \left| \begin{array}{ccc} x_{1k_1} & \cdots & x_{1k_n} \\ \vdots & \ddots & \vdots \\ x_{nk_1} & \cdots & x_{nk_n} \end{array} \right|. \end{aligned} \tag{2.3}$$

**Corollary 2.11.** For  $p = 2$ , the three  $n$ -norms  $\|\cdot, \dots, \cdot\|_2$  in (1.1),  $\|\cdot, \dots, \cdot\|'_2$  in (1.2) and  $\|\cdot, \dots, \cdot\|_{g,2}$  in (2.3) are identical.

For  $p \neq 2$ , we have the following generalization of Theorem 2.4.

**Theorem 2.12.** For every  $x_1, \dots, x_n \in \ell^p$  ( $1 \leq p < \infty$ ), we have

$$(n!)^{\frac{1}{p}-1} \|x_1, \dots, x_n\|_p \leq \|x_1, \dots, x_n\|_{g,p} \leq \|x_1, \dots, x_n\|'_p \leq (n!)^{\frac{1}{p}} \|x_1, \dots, x_n\|_p.$$

*Proof.* For each  $j = 1, \dots, n$ , let  $y_j \in \ell^p$  with  $\|y_j\|_p \leq 1$ . Then take  $u_j = (u_{jk})$  with  $u_{jk} = \|y_j\|_p^{2-p} |y_{jk}|^{p-1} \text{sgn}(y_{jk})$ . We observe that  $u_j \in \ell^{p'}$  with  $\|u_j\|_{p'} = \|y_j\|_p \leq 1$ . As a consequence, we have

$$\|x_1, \dots, x_n\|_{g,p} \leq \|x_1, \dots, x_n\|'_p.$$

By using Theorem 1.2, we obtain

$$\|x_1, \dots, x_n\|_{g,p} \leq \|x_1, \dots, x_n\|'_p \leq (n!)^{\frac{1}{p}} \|x_1, \dots, x_n\|_p.$$

Conversely, suppose that  $\{x_1, \dots, x_n\}$  is a linearly independent set. Using  $x_1^* = x_1$  and so forth as in (2.1), we obtain the left  $g$ -orthogonal set  $\{x_1^*, \dots, x_n^*\}$ . Then, it follows from Theorem 2.2 that

$$(n!)^{\frac{1}{p}-1} \|x_1, \dots, x_n\|_p \leq V(x_1, \dots, x_n) = \|x_1^*\|_p \cdots \|x_n^*\|_p.$$

For  $j = 1, \dots, n$ , let  $y_j = \frac{x_j^*}{\|x_j^*\|_p}$ , so that  $\|y_j\|_p = 1$ . Next, using the properties of matrix determinants and the semi-inner product  $g$ , we have

$$\begin{aligned} \begin{vmatrix} g(y_1, x_1) & \cdots & g(y_n, x_1) \\ \vdots & \ddots & \vdots \\ g(y_1, x_n) & \cdots & g(y_n, x_n) \end{vmatrix} &= \begin{vmatrix} \frac{1}{\|x_1^*\|_p} g(x_1^*, x_1^*) & \cdots & \frac{1}{\|x_n^*\|_p} g(x_n^*, x_1^*) \\ \vdots & \ddots & \vdots \\ \frac{1}{\|x_1^*\|_p} g(x_1^*, x_n^*) & \cdots & \frac{1}{\|x_n^*\|_p} g(x_n^*, x_n^*) \end{vmatrix} \\ &= \|x_1^*\|_p \cdots \|x_n^*\|_p = V(x_1, \dots, x_n) \\ &\geq (n!)^{\frac{1}{p}-1} \|x_1, \dots, x_n\|_p, \end{aligned}$$

whence  $\|x_1, \dots, x_n\|_{g,p} \geq (n!)^{\frac{1}{p}-1} \|x_1, \dots, x_n\|_p$ . Combining with the previous inequalities, we obtain

$$(n!)^{\frac{1}{p}-1} \|x_1, \dots, x_n\|_p \leq \|x_1, \dots, x_n\|_{g,p} \leq \|x_1, \dots, x_n\|'_p \leq (n!)^{\frac{1}{p}} \|x_1, \dots, x_n\|_p.$$

If  $\{x_1, \dots, x_n\}$  is linearly dependent, then all the  $n$ -norms vanish and so we have the equalities. □

**Corollary 2.13.** For  $1 \leq p < \infty$ , the three  $n$ -norms  $\|\cdot, \dots, \cdot\|_{g,p}$ ,  $\|\cdot, \dots, \cdot\|'_p$  and  $\|\cdot, \dots, \cdot\|_p$  are equivalent.

Knowing that the space  $(\ell^p, \|\cdot, \dots, \cdot\|_p)$  is an  $n$ -Banach space in [16], we have a generalization of Corollary 2.6 as follows.

**Corollary 2.14.** For  $1 \leq p < \infty$ , the space  $(\ell^p, \|\cdot, \dots, \cdot\|_{g,p})$  is an  $n$ -Banach space.

### 3. Concluding remarks

In this paper, a new  $n$ -norm is defined using a semi-inner product  $g$  on  $\ell^p$  for  $1 \leq p < \infty$ . Accordingly, on the space  $\ell^p$  ( $1 \leq p < \infty$ ), we have three different  $n$ -norms, namely Gähler's  $n$ -norm  $\|\cdot, \dots, \cdot\|_p$  defined in [8]-[10], Gunawan's  $n$ -norm  $\|\cdot, \dots, \cdot\|_p$  defined in [16], and  $\|\cdot, \dots, \cdot\|_{g,p}$  defined here in (2.3). In Corollary 2.13, we have just seen that the three  $n$ -norms on  $\ell^p$  are equivalent. As expected, the case where  $p = 2$  is special. Here, the three  $n$ -norms on  $\ell^2$  are identical.

In addition to the above three  $n$ -norms, we also have a formula for another  $n$ -norm using the semi-inner product  $g$  on  $\ell^p$  ( $1 \leq p < \infty$ ), namely

$$\|x_1, \dots, x_n\|_{g,p}^\circ = \sup_{\|y_1, \dots, y_n\|_p \leq 1} \begin{vmatrix} g(y_1, x_1) & \cdots & g(y_n, x_1) \\ \vdots & \ddots & \vdots \\ g(y_1, x_n) & \cdots & g(y_n, x_n) \end{vmatrix}.$$

Since  $g(y_j, x_i) = \|y_j\|_p^{2-p} \sum_k |y_{jk}|^{p-1} \text{sgn}(y_{jk}) x_{ik}$ , we obtain

$$\begin{aligned} \|x_1, \dots, x_n\|_{g,p}^\circ &= \left[ \sup_{\|y_1, \dots, y_n\|_p \leq 1} \frac{1}{n!} \prod_{j=1}^n \|y_j\|_p^{2-p} \times \right. \\ &\quad \left. \times \sum_{k_1} \cdots \sum_{k_n} \begin{vmatrix} |y_{1k_1}|^{p-1} \text{sgn}(y_{1k_1}) & \cdots & |y_{1k_n}|^{p-1} \text{sgn}(y_{1k_n}) \\ \vdots & \ddots & \vdots \\ |y_{nk_1}|^{p-1} \text{sgn}(y_{nk_1}) & \cdots & |y_{nk_n}|^{p-1} \text{sgn}(y_{nk_n}) \end{vmatrix} \begin{vmatrix} x_{1k_1} & \cdots & x_{1k_n} \\ \vdots & \ddots & \vdots \\ x_{nk_1} & \cdots & x_{nk_n} \end{vmatrix} \right]. \end{aligned}$$

Note that, for  $p = 2$ , we have  $\|x_1, \dots, x_n\|_{g,2} = \|x_1, \dots, x_n\|_{g,2}^\circ$ . For other values of  $p$ , we can show that

$$\|x_1, \dots, x_n\|_{g,p} \leq (n!)^{2-\frac{1}{p}} \|x_1, \dots, x_n\|_{g,p}^\circ.$$

Indeed, assuming that  $x_1, \dots, x_n$  are linearly independent, let  $x_1^*, \dots, x_n^*$  be the vectors obtained from  $x_1, \dots, x_n$  through the process in (2.1). By taking  $y_j = \frac{x_j^*}{\sqrt{\|x_1^*, \dots, x_n^*\|_p}} (j = 1, \dots, n)$ , we obtain  $\|y_1, \dots, y_n\|_p = 1$ . Next, using the properties of matrix determinants and the semi-inner product  $g$ , we have

$$\begin{aligned} \begin{vmatrix} g(y_1, x_1) & \cdots & g(y_n, x_1) \\ \vdots & \ddots & \vdots \\ g(y_1, x_n) & \cdots & g(y_n, x_n) \end{vmatrix} &= \begin{vmatrix} \frac{1}{\sqrt{\|x_1^*, \dots, x_n^*\|_p}} g(x_1^*, x_1^*) & \cdots & \frac{1}{\sqrt{\|x_1^*, \dots, x_n^*\|_p}} g(x_n^*, x_1^*) \\ \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{\|x_1^*, \dots, x_n^*\|_p}} g(x_1^*, x_n^*) & \cdots & \frac{1}{\sqrt{\|x_1^*, \dots, x_n^*\|_p}} g(x_n^*, x_n^*) \end{vmatrix} \\ &= \frac{\|x_1^*\|_p^2 \cdots \|x_n^*\|_p^2}{\|x_1^*, \dots, x_n^*\|_p}. \end{aligned}$$

Since  $\|x_1, \dots, x_n\|_p \leq (n!)^{1-\frac{1}{p}} \|x_1^*\|_p \cdots \|x_n^*\|_p$  by Theorem 2.2 and  $\|x_1^*, \dots, x_n^*\|_p = \|x_1, \dots, x_n\|_p$ , we obtain

$$\|x_1, \dots, x_n\|_{g,p}^{\circ} \geq (n!)^{\frac{2}{p}-2} \|x_1, \dots, x_n\|_p.$$

Moreover, using Theorem 2.12, we have

$$\|x_1, \dots, x_n\|_{g,p} \leq (n!)^{2-\frac{1}{p}} \|x_1, \dots, x_n\|_{g,p}^{\circ}.$$

It follows from this inequality that the convergence of a sequence in  $\|\cdot, \dots, \cdot\|_{g,p}^{\circ}$  implies the convergence in  $\|\cdot, \dots, \cdot\|_{g,p}$ , and hence also in  $\|\cdot, \dots, \cdot\|_p$ . Unfortunately, up to now, we do not know if the converse is true.

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