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A boundary element method for a class of elliptic boundary value problems of functionally graded media

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A B S T R A C T

A Boundary Element Method (BEM) is derived for obtaining solutions to a class of elliptic boundary value problems (BVPs) of functionally graded media (FGM). Some particular examples are considered to illustrate the application of the BEM.

1. Introduction

Whereas the BEM provides an effective numerical procedure for the solution of BVPs for homogeneous media the same is not generally true for inhomogeneous media. In the case of the inhomogeneous media, the material is assumed to be a functionally graded material, i.e., the material properties vary spatially according to known smooth functions. BVPs for such media have governing equations with variable coefficients. A BEM for 2D diffusion-convection problems in homogeneous anisotropic media has been recently considered by Haddade, Salam, Khaeruddin and Azis in [17]. In recent years some progress toward finding numerical solutions to BVPs for FGM by using BEM has been made. Clements [2], Cheng [4,5], Rangogni [6], Shaw [7], Gipson et al. [8], Ang et al. [9], and Clements and Azis [10] considered the case for isotropic FGM.

In the case of anisotropic FGM there are few published studies. BVPs which are relevant for certain classes of problems for anisotropic FGM have been considered by Azis and Clements [11], Azis et al. [12], Azis and Clements [13], Azis and Clements [14,15]. An elliptic equation which is also relevant for a certain class of problems for anisotropic FGM has been considered by Clements and Rogers [1]. They obtained a boundary integral equation for the case when the coefficients in the equation depend on one spatial variable only. Specifically the equation considered by Clements and Rogers [1] takes the form

$$\frac{\partial}{\partial x_i}\left[\lambda_{ij}(x_2)\,\frac{\partial\phi(x_1,x_2)}{\partial x_j}\right]=0$$

where the coefficients λ_{ij} depend on x_2 only and the repeated summation convention (summing from 1 to 2) is employed.

This paper is concerned with obtaining boundary integral equations for the solution of BVPs governed by equations of the form

$$\frac{\partial}{\partial x_i}\left[\lambda_{ij}(x_1,x_2)\,\frac{\partial\phi(x_1,x_2)}{\partial x_j}\right]=0 \tag{1}$$

Equations of this type govern the behavior of a wide class of BVPs of both isotropic and anisotropic FGM. Antiplane strain in elastostatics and plane thermostatics for anisotropic FGM are two areas for which the governing equation is of the type (1).

Several techniques will be considered for obtaining boundary integral equations for the solution of (1). For each technique it is necessary to place some constraint on the class of coefficients λ_{ij} for which the solution obtained is valid. Some numerical examples are considered to illustrate the application of the boundary integral equations. The analysis of this paper is purely formal; the main aim being to construct effective BEMs for classes of equations which fall within the type (1).

2. The boundary value problem

Referred to a Cartesian frame Ox_1x_2 a solution to (1) is sought which is valid in a region Ω in R^2 with boundary $\partial\Omega$ which consists of a finite number of piecewise smooth closed curves. On $\partial\Omega_1$ the dependent variable $\phi(\mathbf{x})\left(\mathbf{x}=(x_1,x_2)\right)$ is specified and on $\partial\Omega_2$

$$P(\mathbf{x})=\lambda_{ij}\left(\partial\phi/\partial x_j\right)n_i \tag{2}$$

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is specified where $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$ and $\mathbf{n} = (n_1, n_2)$ denotes the outward pointing normal to $\partial\Omega$.

For all points in Ω the matrix of coefficients $[\lambda_{ij}]$ is a real symmetric positive definite matrix so that throughout Ω Eq. (1) is a second order elliptic partial differential equation. Further, the coefficients λ_{ij} are required to be twice differentiable functions of the two independent variables x_1 and x_2.

The method of solution will be to obtain boundary integral equations from which numerical values of the dependent variables ϕ and P may be obtained for all points in Ω. The analysis here is specially relevant to an anisotropic medium but it equally applies to isotropic media. For isotropy, the coefficients in (1) take the form $\lambda_{11} = \lambda_{22}$ and $\lambda_{12} = 0$ and use of these equations in the following analysis immediately yields the corresponding results for an isotropic medium.

3. Reduction to a constant coefficient equation

The coefficients λ_{ij} are required to take the form

$$\lambda_{ij}(\mathbf{x}) = \lambda_{ij}^{(0)}\ g(\mathbf{x}) \tag{3}$$

where the $\lambda_{ij}^{(0)}$ are constants and g is a differentiable function of x. Use of (3) in (1) yields

$$\lambda_{ij}^{(0)} \frac{\partial}{\partial x_i}\left(g \frac{\partial \phi}{\partial x_j}\right) = 0 \tag{4}$$

Let

$$\psi(\mathbf{x}) = g^{1/2}(\mathbf{x})\ \phi(\mathbf{x}) \tag{5}$$

so that (4) may be written in the form

$$\lambda_{ij}^{(0)} \frac{\partial}{\partial x_i}\left[g \frac{\partial\left(g^{-1/2}\psi\right)}{\partial x_j}\right] = 0$$

That is

$$\lambda_{ij}^{(0)}\left[\left(\frac{1}{4}g^{-3/2}\frac{\partial g}{\partial x_i}\frac{\partial g}{\partial x_j} - \frac{1}{2}g^{-1/2}\frac{\partial^2 g}{\partial x_i \partial x_j}\right)\psi + g^{1/2}\frac{\partial^2 \psi}{\partial x_i \partial x_j}\right] = 0 \tag{6}$$

Use of the identity

$$\frac{\partial^2 g^{1/2}}{\partial x_i \partial x_j} = -\frac{1}{4}g^{-3/2}\frac{\partial g}{\partial x_i}\frac{\partial g}{\partial x_j} + \frac{1}{2}g^{-1/2}\frac{\partial^2 g}{\partial x_i \partial x_j}$$

permits (6) to be written in the form

$$g^{1/2}\lambda_{ij}^{(0)}\frac{\partial^2\psi}{\partial x_i \partial x_j} - \psi\,\lambda_{ij}^{(0)}\frac{\partial^2 g^{1/2}}{\partial x_i \partial x_j} = 0$$

It follows that if g is such that

$$\lambda_{ij}^{(0)}\frac{\partial^2 g^{1/2}}{\partial x_i \partial x_j} + k g^{1/2} = 0 \tag{7}$$

then the transformation (5) carries the variable coefficients Eq. (4) to the constant coefficients equation

$$\lambda_{ij}^{(0)}\frac{\partial^2\psi}{\partial x_i \partial x_j} + k\psi = 0 \tag{8}$$

where k is a constant.

Also, substitution of (3) and (5) into (2) gives

$$P = -P^{[g]}\psi + P^{[\psi]}g^{1/2} \tag{9}$$

where

$$P^{[g]}(\mathbf{x}) = \lambda_{ij}^{(0)}\frac{\partial g^{1/2}}{\partial x_j}n_i \qquad P(\mathbf{x}) = \lambda_{ij}^{(0)}\frac{\partial \psi}{\partial x_j}n_i$$

A boundary integral equation for the solution of (8) is given in Clements [3] in the form

$$\eta(\mathbf{x}_0)\ \psi(\mathbf{x}_0) = \int_{\partial\Omega}\left[\Gamma(\mathbf{x},\mathbf{x}_0)\ \psi(\mathbf{x}) - \Phi(\mathbf{x},\mathbf{x}_0)\ P^{[\psi]}(\mathbf{x})\right]ds(\mathbf{x}) \tag{10}$$

where $\mathbf{x}_0 = (a,b)$, $\eta = 0$ if $(a,\ b)\notin\Omega\cup\partial\Omega$, $\eta = 1$ if $(a,\ b)\in\Omega$, $\eta = \frac{1}{2}$ if $(a,\ b)\in\partial\Omega$ and $\partial\Omega$ has a continuously turning tangent at $(a,\ b)$.

The so called fundamental solution Φ in (10) is any solution of the equation

$$\lambda_{ij}^{(0)}\frac{\partial^2\Phi}{\partial x_i \partial x_j} + k\Phi = \delta(\mathbf{x} - \mathbf{x}_0)$$

and the Γ is given by

$$\Gamma(\mathbf{x},\mathbf{x}_0) = \lambda_{ij}^{(0)}\frac{\partial\Phi(\mathbf{x},\mathbf{x}_0)}{\partial x_j}n_i$$

where δ is the Dirac delta function. For two-dimensional problems Φ and Γ are given by

$$\Phi(\mathbf{x},\mathbf{x}_0) = \begin{cases}\frac{K}{2\pi}\ln R & \text{if } k = 0\\ \frac{\imath K}{4}H_0^{(2)}(\omega R) & \text{if } k > 0\\ \frac{-K}{2\pi}K_0(\omega R) & \text{if } k < 0\end{cases}$$

$$\Gamma(\mathbf{x},\mathbf{x}_0) = \begin{cases}\frac{K}{2\pi}\frac{1}{R}\lambda_{ij}^{(0)}\frac{\partial R}{\partial x_j}n_i & \text{if } k = 0\\ \frac{-\imath K\omega}{4}H_1^{(2)}(\omega R)\lambda_{ij}^{(0)}\frac{\partial R}{\partial x_j}n_i & \text{if } k > 0\\ \frac{K\omega}{2\pi}K_1(\omega R)\lambda_{ij}^{(0)}\frac{\partial R}{\partial x_j}n_i & \text{if } k < 0\end{cases} \tag{11}$$

where

$$\begin{aligned} K &= \ddot{\tau}/\zeta\\ \omega &= \sqrt{|k|/\zeta}\\ \zeta &= [\lambda_{11}^{(0)} + \lambda_{12}^{(0)}(\tau + \overline{\tau}) + \lambda_{22}^{(0)}\tau\overline{\tau}]/2\\ R &= \sqrt{(\dot{x}_1 - \dot{a})^2 + (\dot{x}_2 - \dot{b})^2}\\ \dot{x}_1 &= x_1 + \dot{\tau}x_2\\ \dot{a} &= a + \dot{\tau}b\\ \dot{x}_2 &= \ddot{\tau}x_2\\ \dot{b} &= \ddot{\tau}b \end{aligned}$$

where $\dot{\tau}$ and $\ddot{\tau}$ are respectively the real and the positive imaginary parts of the complex root τ of the quadratic

$$\lambda_{11}^{(0)} + 2\lambda_{12}^{(0)}\tau + \lambda_{22}^{(0)}\tau^2 = 0$$

and $H_0^{(2)}$, $H_1^{(2)}$ denote the Hankel function of second kind and order zero and order one respectively. K_0, K_1 denote the modified Bessel function of order zero and order one respectively, $\imath$ represents the square root of minus one and the bar denotes the complex conjugate. A technique for finding the fundamental solution Φ in Eq. (11) may be found in Azis [16].

The derivatives $\partial R/\partial x_j$ needed for the calculation of the Γ in (11) are given by

$$\begin{aligned}\frac{\partial R}{\partial x_1} &= \frac{1}{R}(\dot{x}_1 - \dot{a})\\ \frac{\partial R}{\partial x_2} &= \dot{\tau}\left[\frac{1}{R}(\dot{x}_1 - \dot{a})\right] + \ddot{\tau}\left[\frac{1}{R}(\dot{x}_2 - \dot{b})\right]\end{aligned}$$

Use of (5) and (9) in (10) yields

$$\eta(\mathbf{x}_0)\ g^{1/2}(\mathbf{x}_0)\ \phi(\mathbf{x}_0) = \int_{\partial\Omega}\Big\{\left[g^{1/2}(\mathbf{x})\ \Gamma(\mathbf{x},\mathbf{x}_0) - P^{[g]}(\mathbf{x})\ \Phi(\mathbf{x},\mathbf{x}_0)\right]\phi(\mathbf{x}) - \left[g^{-1/2}(\mathbf{x})\ \Phi(\mathbf{x},\mathbf{x}_0)\right]P(\mathbf{x})\Big\}ds(\mathbf{x})$$

This equation provides a boundary integral equation for determining ϕ and P at all points of Ω.

The analysis of the section requires that the coefficients λ_{ij} are of the form (3) with g satisfying (7). This condition on g allows for considerable choice in the coefficients. For example, when $k = 0$, g can assume a number of multiparameter forms with the parameters being employed to fit λ_{ij} to numerical data for the coefficients. Possible multiparameter forms include

$$g(\mathbf{x}) = \left(\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2\right)^2$$
$$g(\mathbf{x}) = \left[\Re\{\alpha_0 + \alpha_1 z + \alpha_2 z^2 + \cdots + \alpha_n z^n\}\right]^2$$

where the α_i are constants, $\Re$ denotes the real part of a complex number and $z = x_1 + \tau x_2$. More generally, the square of the real part of any analytical function of the complex variable z can serve as a possible form for g. For the case when $k \neq 0$ some possible multiparameter forms of g are

$$g(\mathbf{x}) = \left[A\cos(\alpha_m x_m)\right]^2 \quad \text{with} \quad \lambda_{ij}^{(0)}\alpha_i\alpha_j = k \tag{12}$$

$$g(\mathbf{x}) = \left[A\exp(\alpha_m x_m)\right]^2 \quad \text{with} \quad \lambda_{ij}^{(0)}\alpha_i\alpha_j = -k \tag{13}$$

where A, α_m are real constants.

4. Further reduction to constant coefficients

The analysis of this paper has so far been concerned with coefficients which fall within the general class given by (3). The following analysis seeks to consider the case when the coefficients $\lambda_{ij}(\mathbf{x})$ are not all proportional to the same function of $\mathbf{x}$ and thus fall outside the general class given by (3).

Attention will be restricted to anisotropic media for which the media have symmetry properties which lead to the coefficient λ_{12} being zero. In this case Eq. (1) becomes

$$\frac{\partial}{\partial x_1}\left[\lambda_{11}(x_1,x_2)\frac{\partial\phi}{\partial x_1}\right] + \frac{\partial}{\partial x_2}\left[\lambda_{22}(x_1,x_2)\frac{\partial\phi}{\partial x_2}\right] = 0 \tag{14}$$

It will be further assumed that the coefficients λ_{11} and λ_{22} have variable separable forms so that they can be written in the form

$$\lambda_{11}(x_1,x_2) = X_1(x_1)Y_1(x_2) \tag{15}$$

$$\lambda_{22}(x_1,x_2) = X_2(x_1)Y_2(x_2) \tag{16}$$

Substitution of (15) and (16) into (14) yields

$$\frac{1}{X_2(x_1)}\frac{\partial}{\partial x_1}\left[X_1(x_1)\frac{\partial\phi}{\partial x_1}\right] + \frac{1}{Y_1(x_2)}\frac{\partial}{\partial x_2}\left[Y_2(x_2)\frac{\partial\phi}{\partial x_2}\right] = 0$$

The new independent variables

$$\xi_1 = \int\left(\frac{X_2(x_1)}{X_1(x_1)}\right)^{1/2} dx_1 \qquad \xi_2 = \int\left(\frac{Y_1(x_2)}{Y_2(x_2)}\right)^{1/2} dx_2 \tag{17}$$

now provide

$$\frac{1}{M(\xi_1)}\frac{\partial}{\partial\xi_1}\left[M(\xi_1)\frac{\partial\phi}{\partial\xi_1}\right] + \frac{1}{N(\xi_2)}\frac{\partial}{\partial\xi_2}\left[N(\xi_2)\frac{\partial\phi}{\partial\xi_2}\right] = 0 \tag{18}$$

where

$$M = \left(X_2X_1\right)^{1/2} \qquad N = \left(Y_2Y_1\right)^{1/2} \tag{19}$$

A new dependent variable ψ is now introduced according to

$$\phi = M^{-1/2}N^{-1/2}\psi \tag{20}$$

Use of (20) in (18) yields

$$\nabla^2\psi - \left[\left(M^{-1/2}N^{-1/2}\right)\nabla^2\left(M^{1/2}N^{1/2}\right)\right]\psi = 0 \tag{21}$$

where

$$\nabla^2 \equiv \partial^2/\partial{\xi_1}^2 + \partial^2/\partial{\xi_2}^2$$

If

$$\nabla^2\left(M^{1/2}N^{1/2}\right) = 0 \tag{22}$$

then from (21) ψ satisfies Laplace's equation

$$\nabla^2\psi = 0$$

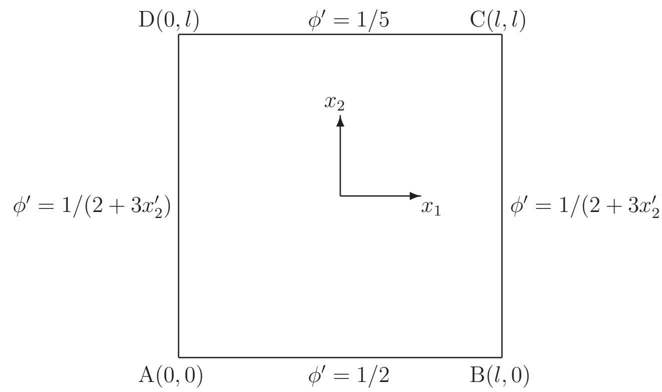

Fig. 1. The boundary conditions for Example 5.1 .

Since M is a function of ξ_1 only and N is a function of ξ_2 only then in (22) M and N must adopt the forms

$$M(\xi_1) = \left(\alpha\xi_1 + \beta\right)^2 \qquad N(\xi_2) = \left(\gamma\xi_2 + \delta\right)^2$$

where α, β, γ and δ are constants.

The boundary integral equations for ψ in the new frame $O\xi_1\xi_2$ may be obtained from (9) and (10) with $\lambda_{11}^{(0)} = \lambda_{22}^{(0)} = 1, \lambda_{12}^{(0)} = 0$. Specifically the equation for ψ is

$$\eta(\xi_0)\,\psi(\xi_0) = \int_{\partial\Omega_\xi}\left[\Gamma(\xi,\xi_0)\,\psi(\xi) - \Phi(\xi,\xi_0)\,P^{[\psi]}(\xi)\right]ds(\xi)$$

where Φ and Γ are given by (11) with $k = 0$ and the region Ω_ξ with boundary $\partial\Omega_\xi$ denoting the domain and boundary under consideration referred to the $O\xi_1\xi_2$ frame.

The boundary integral equation for ϕ is given by

$$\eta(\xi_0)\,g^{1/2}(\xi_0)\,\phi(\xi_0) = \int_{\partial\Omega_\xi}\left\{\left[g^{1/2}(\xi)\,\Gamma(\xi,\xi_0) - P^{[g]}(\xi)\,\Phi(\xi,\xi_0)\right]\phi(\xi) - \left[g^{-1/2}(\xi)\,\Phi(\xi,\xi_0)\right]P(\xi)\right\}ds(\xi) \tag{23}$$

where

$$g(\xi) = M(\xi_1)N(\xi_2)$$

5. Numerical examples

In this section some particular BVPs are solved numerically by employing the integral equations obtained in Sections 3 and 4. In implementing this method to obtain numerical results standard boundary element procedure is employed (see for example Clements [3]).

5.1. Example 5.1

Consider a problem governed by an equation of the type (14) for a medium of geometry as shown in Fig. 1 with the coefficients of the forms (15) and (16).

Specifically the problem is governed by

$$\frac{\partial}{\partial x_1}\left[\lambda_{11}(x_1,x_2)\frac{\partial\phi}{\partial x_1}\right] + \frac{\partial}{\partial x_2}\left[\lambda_{22}(x_1,x_2)\frac{\partial\phi}{\partial x_2}\right] = 0$$

with coefficients

$$\lambda'_{11}(x_1,x_2) = (1 + 2x'_1)^{3/2}(2 + 3x'_2)^2$$
$$\lambda'_{22}(x_1,x_2) = (1 + 2x'_1)^{1/2}(2 + 3x'_2)^2$$

The boundary conditions are (see Fig. 1)

$\phi' = 1/2$ on AB
$\phi' = 1/(2 + 3x'_2)$ on BC
$\phi' = 1/5$ on CD
$\phi' = 1/(2 + 3x'_2)$ on AD

Table 1
Numerical and analytical results for Example 5.1.

Position (x_1', x_2')	ϕ' BEM 8 segments	ϕ' BEM 16 segments	ϕ' BEM 32 segments	ϕ' Analytical
(0.9,0.4)	0.285323	0.313721	0.312487	0.312500
(0.2,0.1)	0.431316	0.434713	0.434756	0.434783
(0.6,0.3)	0.344864	0.344839	0.344834	0.344828
(0.8,0.9)	0.207950	0.212652	0.212752	0.212766
(0.4,0.6)	0.263112	0.263144	0.263153	0.263158
(0.9,0.2)	0.396525	0.381003	0.384669	0.384615
(0.5,0.1)	0.433516	0.434958	0.434778	0.434783
(0.3,0.8)	0.226931	0.227248	0.227264	0.227273
(0.9,0.9)	0.185691	0.213604	0.212776	0.212766

For this particular problem the coefficients λ_{11} and λ_{22} are of the type (15) and (16) with

$$X_1'(x_1) = (1 + 2x_1')^{3/2} \qquad Y_1'(x_2) = (2 + 3x_2')^2$$

$$X_2'(x_1) = (1 + 2x_1')^{1/2} \qquad Y_2'(x_2) = (2 + 3x_2')^2$$

where $X_1' = X_1/\hat{X}_1$, $X_2' = X_2/\hat{X}_2$, $Y_1' = Y_1\hat{X}_1/\hat{\lambda}$, $Y_2' = Y_2\hat{X}_2/\hat{\lambda}$, $\hat{X}_1$, $\hat{X}_2$ are reference values of X_1 and X_2 respectively. Thus from (17) and (19)

$$M' = 1 + 2x_1' = (\xi_1')^2 \qquad N' = (2 + 3x_2')^2 = (2 + 3\xi_2')^2$$

where, $M' = M/\widehat{M}$, $\widehat{M} = (\hat{X}_1\hat{X}_2)^{1/2}$, $N' = N/\hat{N}$, $\hat{N} = \hat{\lambda}/(\hat{X}_1\hat{X}_2)$, $\xi_i' = \xi_i/\hat{\xi}$ (for $i = 1, 2$), $\hat{\xi} = (\hat{X}_1\hat{X}_2)^{1/2} l$. The M' and N' satisfy Eq. (22). It follows from (20) that the transformation

$$\begin{aligned}\phi' &= (M')^{-1/2}(N')^{-1/2}\psi' \\ &= (\xi_1')^{-1}(2 + 3\xi_2')^{-1}\psi'\end{aligned}$$

where $\psi' = \psi/(\hat{\phi}\hat{\lambda}^{1/2})$, transforms the original partial differential equation to

$$\nabla^2\psi' = 0$$

where $\nabla^2 \equiv \partial^2/\partial\xi_1'^2 + \partial^2/\partial\xi_2'^2$.

Table 1 shows a comparison between BEM results obtained using Eq. (23) and analytical results for some interior points. The analytical solution to this problem is $\phi' = 1/(2 + 3x_2')$.

5.2. Example 5.2

Consider the analytical solution to (1)

$$\phi' = \frac{\sin \beta' x_2'}{\cos(\alpha_1' x_1' + \alpha_2' x_2')} \tag{24}$$

where $\alpha_i' = \alpha_i l$ $(i = 1, 2)$, α_i is given in (12) and β' is a dimensionless constant, to a problem for an inhomogeneous material occupying the region of a unit square depicted in Fig. 2.

The coefficients of the material vary with position in the square according to

$$\left[\lambda_{ij}'\right] = \begin{bmatrix} \cos^2(\alpha_1' x_1' + \alpha_2' x_2') & 0.5\cos^2(\alpha_1' x_1' + \alpha_2' x_2') \\ 0.5\cos^2(\alpha_1' x_1' + \alpha_2' x_2') & 0.5\cos^2(\alpha_1' x_1' + \alpha_2' x_2') \end{bmatrix} \tag{25}$$

The boundary conditions are (see Fig. 2)

P' (as may be obtained from (24)) is known on AB, BC and CD
ϕ' (as may be calculated from (24)) is known on AD

The coefficients (25) may be written in the form (3) with

$$\left[\lambda_{ij}'^{(0)}\right] = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

$$g(\mathbf{x}) = \cos^2(\alpha_1' x_1' + \alpha_2' x_2')$$

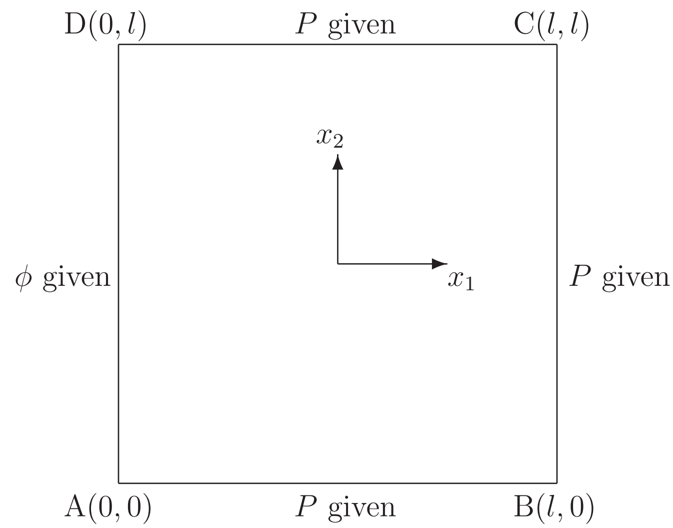

Fig. 2. The boundary conditions for Example 5.2 and Example 5.3 .

Table 2
BEM and analytical solutions for Example 5.2.

Position	BEM			Analytical		
(x_1', x_2')	ϕ'	$\partial\phi'/\partial x_1'$	$\partial\phi'/\partial x_2'$	ϕ'	$\partial\phi'/\partial x_1'$	$\partial\phi'/\partial x_2'$
			40 segments			
(0.3,0.5)	0.5064	0.0953	0.9886	0.5073	0.0877	0.9928
(0.5,0.5)	0.5290	0.1322	1.0475	0.5282	0.1221	1.0562
(0.7,0.5)	0.5596	0.1752	1.1274	0.5566	0.1642	1.1391
(0.9,0.5)	0.6031	0.3528	1.1864	0.5946	0.2182	1.2482
(0.5,0.1)	0.1107	0.0307	1.0296	0.1041	0.0153	1.0485
(0.5,0.3)	0.3190	0.0725	1.0488	0.3157	0.0594	1.0642
(0.5,0.7)	0.7362	0.2103	1.0199	0.7366	0.2042	1.0240
(0.5,0.9)	0.9350	0.3116	0.9500	0.9361	0.3063	0.9671
			80 segments			
(0.3,0.5)	0.5068	0.0913	0.9912	0.5073	0.0877	0.9928
(0.5,0.5)	0.5285	0.1267	1.0526	0.5282	0.1221	1.0562
(0.7,0.5)	0.5580	0.1693	1.1340	0.5566	0.1642	1.1391
(0.9,0.5)	0.5968	0.2119	1.2446	0.5946	0.2182	1.2482
(0.5,0.1)	0.1072	0.0231	1.0389	0.1041	0.0153	1.0485
(0.5,0.3)	0.3171	0.0658	1.0571	0.3157	0.0594	1.0642
(0.5,0.7)	0.7364	0.2073	1.0223	0.7366	0.2042	1.0240
(0.5,0.9)	0.9357	0.3088	0.9659	0.9361	0.3063	0.9671
			160 segments			
(0.3,0.5)	0.5071	0.0896	0.9921	0.5073	0.0877	0.9928
(0.5,0.5)	0.5284	0.1244	1.0545	0.5282	0.1221	1.0562
(0.7,0.5)	0.5573	0.1668	1.1365	0.5566	0.1642	1.1391
(0.9,0.5)	0.5959	0.2207	1.2448	0.5946	0.2182	1.2482
(0.5,0.1)	0.1057	0.0195	1.0433	0.1041	0.0153	1.0485
(0.5,0.3)	0.3165	0.0628	1.0606	0.3157	0.0594	1.0642
(0.5,0.7)	0.7366	0.2059	1.0232	0.7366	0.2042	1.0240
(0.5,0.9)	0.9359	0.3077	0.9663	0.9361	0.3063	0.9671

Thus, $g(\mathbf{x})$ takes the form (12) with $A = 1$. The parameters k' $(k' = kl^2/\hat{\lambda})$ and α_1' are chosen to be 0.5 and the parameter α_2' is required to satisfy the condition in (12) $(\lambda_{ij}'^{(0)}\alpha_i'\alpha_j' = k')$. Specifically, α_2' satisfies

$$\alpha_2' = \frac{1}{\lambda_{22}'^{(0)}}\left[-\lambda_{12}'^{(0)}\alpha_1' + \sqrt{\lambda_{12}'^{(0)2}\alpha_1'^2 - \lambda_{22}'^{(0)}\left(\lambda_{11}'^{(0)}\alpha_1'^2 - k'\right)}\right]$$

The parameter β' is defined as $\beta' = \sqrt{k'/\lambda_{22}'^{(0)}}$.

Table 2 shows a comparison between the BEM and the analytical solutions. The BEM solutions converge to the analytical solutions as the number of segments increases.

Table 3
BEM and analytical solutions for Example 5.3.

Position	BEM			Analytical		
(x'_1, x'_2)	ϕ'	$\partial\phi'/\partial x'_1$	$\partial\phi'/\partial x'_2$	ϕ'	$\partial\phi'/\partial x'_1$	$\partial\phi'/\partial x'_2$
			40 segments			
(0.3,0.5)	0.9631	0.4751	−0.3487	0.9675	0.4838	−0.3541
(0.5,0.5)	1.0637	0.5305	−0.3921	1.0693	0.5346	−0.3914
(0.7,0.5)	1.1755	0.5886	−0.4356	1.1817	0.5909	−0.4325
(0.9,0.5)	1.3064	0.8583	−0.4897	1.3060	0.6530	−0.4780
(0.5,0.1)	1.2329	0.6192	−0.4397	1.2379	0.6189	−0.4531
(0.5,0.3)	1.1454	0.5717	−0.4240	1.1505	0.5752	−0.4211
(0.5,0.7)	0.9883	0.4909	−0.3646	0.9938	0.4969	−0.3638
(0.5,0.9)	0.9172	0.4580	−0.3572	0.9236	0.4618	−0.3381
			80 segments			
(0.3,0.5)	0.9655	0.4807	−0.3550	0.9675	0.4838	−0.3541
(0.5,0.5)	1.0666	0.5321	−0.3914	1.0693	0.5346	−0.3914
(0.7,0.5)	1.1786	0.5889	−0.4330	1.1817	0.5909	−0.4325
(0.9,0.5)	1.3023	0.6306	−0.4774	1.3060	0.6530	−0.4780
(0.5,0.1)	1.2353	0.6183	−0.4536	1.2379	0.6189	−0.4531
(0.5,0.3)	1.1479	0.5732	−0.4214	1.1505	0.5752	−0.4211
(0.5,0.7)	0.9911	0.4949	−0.3648	0.9938	0.4969	−0.3638
(0.5,0.9)	0.9207	0.4590	−0.3391	0.9236	0.4618	−0.3381
			160 segments			
(0.3,0.5)	0.9665	0.4822	−0.3542	0.9675	0.4838	−0.3541
(0.5,0.5)	1.0679	0.5333	−0.3911	1.0693	0.5346	−0.3914
(0.7,0.5)	1.1801	0.5897	−0.4324	1.1817	0.5909	−0.4325
(0.9,0.5)	1.3042	0.6520	−0.4782	1.3060	0.6530	−0.4780
(0.5,0.1)	1.2364	0.6191	−0.4537	1.2379	0.6189	−0.4531
(0.5,0.3)	1.1490	0.5740	−0.4204	1.1505	0.5752	−0.4211
(0.5,0.7)	0.9924	0.4958	−0.3641	0.9938	0.4969	−0.3638
(0.5,0.9)	0.9222	0.4604	−0.3387	0.9236	0.4618	−0.3381

5.3. *Exayes mple 5.3*

Consider the analytical solution to (1)

$$\phi' = \frac{\exp \beta' x'_2}{\exp(\alpha'_1 x'_1 + \alpha'_2 x'_2)}$$

where $\alpha'_i = \alpha_i l$ $(i = 1, 2)$, α_i is given in (13) and β' is a dimensionless constant, for a problem which is associated with an inhomogeneous medium as shown in Fig. 2 with coefficients

$$\left[\lambda'_{ij}\right] = \begin{bmatrix} \exp[2(\alpha'_1 x'_1 + \alpha'_2 x'_2)] & 0.5\exp[2(\alpha'_1 x'_1 + \alpha'_2 x'_2)] \\ 0.5\exp[2(\alpha'_1 x'_1 + \alpha'_2 x'_2)] & 0.5\exp[2(\alpha'_1 x'_1 + \alpha'_2 x'_2)] \end{bmatrix} \tag{26}$$

The boundary conditions are (see Fig. 2)

P' is given on AB, BC and CD
ϕ' is given on AD

The coefficients (26) may be written in the form (3) with

$$\left[\lambda'^{(0)}_{ij}\right] = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$
$$g(\mathbf{x}) = \exp[2(\alpha'_1 x'_1 + \alpha'_2 x'_2)]$$

so that $g(\mathbf{x})$ takes the form (13) with $A = 1$. The parameters k' and α'_1 are chosen to be –0.5 and the parameter α'_2 is required to satisfy the condition in (13) ($\lambda'^{(0)}_{ij}\alpha'_i\alpha'_j = -k'$). Specifically, α'_2 satisfies

$$\alpha'_2 = \frac{1}{\lambda'^{(0)}_{22}}\left[-\lambda'^{(0)}_{12}\alpha'_1 + \sqrt{\lambda'^{(0)2}_{12}\alpha'^2_1 - \lambda'^{(0)}_{22}\left(\lambda'^{(0)}_{11}\alpha'^2_1 + k'\right)}\right]$$

The parameter β' is defined as $\beta' = \sqrt{-k'/\lambda'^{(0)}_{22}}$.

Again, Table 3 shows that the BEM solutions converge to the analytical solutions as the number of segments increases.

6. Conclusion

Some BEMs are obtained for a class of two dimensional elliptic BVPs for FGM. The methods can be applied to a variety of problems in such areas as antiplane strain in elastostatics and plane thermostatic problems for anisotropic FGM.

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