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THE EXISTENCE OF GLOBAL ATTRACTOR IN THE LORENZ SYSTEM

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Abstract

The behavior of dynamic systems in long term dynamics can be described by the global attractor. Let V be a metric space. Then the global attractor is a nonempty, compact, and invariant set of A subset V which attracts every bounded subset of V. This paper analyzes the existence of global attractor in the Lorenz system. At the initial stage, we prove the solutions of the Lorenz system bounded for t approaching infinity. Afterwards, we take bounded sets $\mathfrak{B}_i$ of $B(0, R)$ and choose an open set $B(0, \rho)$ in norm space containing the bounded solutions. We then operate a strongly continuous semigroup $\{T(t)\}$ to any bounded set $\mathfrak{B}_i$ as well as to the set $B(0, \rho)$. The result of this operation will be identified as an attractor if and only if any $T(t)\mathfrak{B}_i$ is absorbed by $B(0, \rho)$. The intersection of the closure union $T(t)\mathfrak{B}_i$ is called the ω-limit set of B. If the ω-limit set is a compact attractor, then the existence of global attractor for the Lorenz system is proved.

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1. Introduction

Most of problems in dynamical systems arising from mathematical physics are generated by partial differential equations and semigroup in infinite dimensional space. In this paper, we consider the Lorenz system originally described by partial differential equations and simplified to three nonlinear differential equations

$$\frac{dx}{dt} = -\sigma x + \sigma y,$$

$$\frac{dy}{dt} = rx - y - xz,$$

$$\frac{dz}{dt} = -bz + xy, \tag{1.1}$$

where x represents the velocity, y and z the temperature of the fluid and r, σ, b are the positive parameters determined by the heating of the fluid, the physical properties of the fluid and the height of the layer, respectively. The peculiar result of the Lorenz systems is that they produce deterministic chaos. It is chaotic because even small changes in the initial conditions can lead to very different behavior over long time period [4, 5]. Since Edward N. Lorenz discovered the Lorenz attractor (also known as the butterfly attractor), great achievements have been made in the area of nonlinear systems.

The fundamental properties of chaotic systems are such that the existence of the global attractor plays a very important role in further research on chaos. The global attractor is a compact, invariant set which absorbs every bounded set in the system. This concept can describe the asymptotic behaviors of dynamical systems, [6, 8-10]. By using the generalized Lyapunov function, a simple proof of the globally attractive and positive invariant set of the Lorenz system with ultimate bound has been reported in [2, 3, 11]. Based on the concept of compactness called ω-*limit compact* in a semigroup as in Yu and Liao [11], Ma et al. [7], we show in this paper that there exists a global attractor for a strongly continuous semigroup C^0 to the equations system (1.1), if and only if

(1) there is an absorbing set, and

(2) the semigroup is ω-limit compact.

Firstly, we need to introduce some notations and definitions. Let $X = (x, y, z)$ and $\Omega \subseteq R^3$ be a compact (bounded and closed) set containing the origin. It is known that (1.1) has a unique bounded solution, [2, 3, 9]. This allows us to define the semigroup of operators.

Definition 1.1. Let V be a metric space. The dynamical system in V is described by a family of operators $\{T(t)\}$ of maps V into itself. The family is called a C^0 *semigroup* if it satisfies

(i) $T(0) = I$, identity in V,

(ii) $T(t + s) = T(t)T(s)$,

(iii) the function $T(t)x$ from $[0, \infty) \times V$ to V is continuous at each point $(t, x) \in [0, \infty) \times V$.

A is an invariant set for semigroup $\{T(t)\}$ if

$$T(t)A = A \text{ for } t \geq 0.$$

Lemma 1.1. *If* $\{T(t)\}$ *is a compact* C^0 *semigroup on metric space* V, *then*

$$\forall B \subset V, \ \forall \tau_2 > \tau_1 > 0, \ \bigcup\nolimits_{t \in [\tau_1, \tau_2]} T(t)B$$

is bounded in V.

Definition 1.2. Let B be a subset of V and U be an open subset containing B. Then B is said to be *absorbed* in the set U if the orbit of each limited subset of U in B after certain conditions.

(i) for every $\mathcal{B}_0 \subset U$, $\mathcal{B}_0$ is bounded,

(ii) there exists $t_{\mathcal{B}_0}$ such that

$$T(t)\mathcal{B}_0 \subset B, \quad \forall t \geq t_{\mathcal{B}_0}$$

B can also to be absorbed in all finite subsets of V.

Definition 1.3. Let B_1, B_2 be two subsets of V. Then we say that B_2 is $\{T(t)\}$*-attracted* by B_1 if

$$d(T(t)B_2, B_1) \to 0 \text{ as } t \to \infty \text{ for each } t \geq 0$$

and

$$d(T(t)B_2, B_1) = \sup_{b_2 \in T(t)B_2} \inf_{b_1 \in B_1} dist_V(b_1, b_2).$$

Corollary 1.1. *Let $\{T(t)\}$ be a C^0 semigroup on a metric space V. If $\{T(t)\}$ is strongly continuous and asymptotically smooth on V, and the set*

$$\bigcup_{t \geq 0} T(t)B$$

is bounded for some number $t > 0$, then the semigroup $\{T(t)\}$ is compact in V.

Definition 1.4. Let V be a norm space. For any $B \subset V$, positive orbit $\gamma^+(B)$ can be defined as

$$\gamma^+(B) = \bigcup_{t \geq 0} T(t)B.$$

Definition 1.5. Let V be a norm space. Let $B \subset V$. Then ω*-limit set* of B is

$$\omega(B) = \bigcap_{s \geq 0} cl_V \bigcup_{t \geq s} T(t)B$$

with orbit $\gamma^+(B) = \bigcup_{t \geq 0} T(t)B$, and $cl_V \bigcup_{t \geq s} T(t)B$ is the closure of set $\bigcup_{t \geq s} T(t)B$ in V.

Remark. $\{T(t)\}$ is a completely continuous semigroup for $t > 0$ in the sense of Hale [1], which means the semigroup is compact for each bounded set $B \subset V$ and each number $\tau > 0$ the set $\bigcup_{t \in [0,\, \tau]} T(t)B$ is bounded in V.

The following assumptions are well known in [9].

Assumption A. If $f \in V$, then there exist an absorbing set in V, a constant ρ_0 and time $t_0(|u(0)|)$ such that for the solution to $u(t) = T(t)u(0)$ we have for every $t \geq t_0(|u(0)|)$,

$$t_0(|u(0)|) = \max\left(-\frac{1}{\nu\lambda_1} \ln\left(\frac{\| f \|_*^2}{\nu^2 \lambda_1 | u(0) |} \right),\ 0 \right)$$

with

$$t_0(|u(0)|) \equiv \frac{1}{2\lambda_1} \ln \frac{\lambda_1 | u(0) |^2}{k | \Omega |}$$

2. Main Results

In this section, the existence of the global attractor $A \subset R^3(\Omega)$ for the problem (1.1) is proved under Assumption A. The main result is the following.

Theorem 2.1. *Let norm space* $V = \mathbb{R}^3$ *and* $\{T(t)\}$ *be a strongly continuous semigroup. Under Assumption A, if there exists an open set* $\mathcal{B}_0$ *bounded in B, such that B is absorbed in U, then the* ω*-limit set* $A = \omega(B)$ *in B is a compact attractor and A is a global attractor in U.*

Proof. Let $V = \mathbb{R}^3$, $\boldsymbol{u} = (x, y, z)$. To show that the solution $\boldsymbol{u} = (x, y, z)$ of (1.1) remains bounded at $t \to \infty$ and there is an absorbing set in V, we multiply equation (1.1) sequentially with x, y and z, thus obtaining

$$x\frac{dx}{dt} + \sigma x^2 - \sigma xy = 0, \tag{2.1}$$

$$y\frac{dy}{dt} + \sigma xy + y^2 + xyz = 0, \tag{2.2}$$

$$z\frac{dz}{dt} + bz^2 - xyz = -zb(r + \sigma). \tag{2.3}$$

Summing equations (2.1), (2.2) and (2.3), then differentiating and integrating both sides, we obtain:

$$\frac{1}{2}\frac{d}{dt}(x^2 + y^2 + z^2) + \sigma x^2 + y^2 + bz^2 = -zb(r + \sigma). \tag{2.4}$$

Based on the square of the absolute value, (2.4) becomes

$$\frac{1}{2}\frac{d}{dt}|\boldsymbol{u}|^2 + \sigma x^2 + y^2 + bz^2 = -zb(r + \sigma). \tag{2.5}$$

From Young's inequality with $\varepsilon = 2(b - 1)$, $n = -z$, $m = b(r + \sigma)$, and $p = q = 2$, the right side of the above equation becomes

$$-zb(r + \sigma) \le (b - 1)z^2 + \frac{b^2}{4(b - 1)}(r + \sigma)^2. \tag{2.6}$$

Thus, it follows from (2.5) and (2.6) that

$$\frac{1}{2}\frac{d}{dt}|\boldsymbol{u}|^2 + \sigma x^2 + y^2 + bz^2 \le (b - 1)z^2 + \frac{b^2}{4(b - 1)}(r + \sigma)^2.$$

If $l = \min(1, \sigma)$, then we have

$$\frac{1}{2}\frac{d}{dt}|\boldsymbol{u}|^2 + l\left(\sqrt{x^2 + y^2 + z^2}\right)^2 \le \frac{b^2}{4(b - 1)}(r + \sigma)^2$$

which is equivalent to

$$\frac{1}{2}\frac{d}{dt}|\boldsymbol{u}|^2 + l|\boldsymbol{u}|^2 \le \frac{b^2}{4(b - 1)}(r + \sigma)^2,$$

$$\frac{d}{dt}|\boldsymbol{u}|^2 + 2l|\boldsymbol{u}|^2 \le \frac{b^2}{2(b - 1)}(r + \sigma)^2.$$

Computing the differential equations respect to t gives the following

$$e^{2lt}\frac{d}{dt}|\boldsymbol{u}|^2 + 2le^{2lt}|\boldsymbol{u}|^2 \le \frac{b^2}{2(b-1)}(r+\sigma)^2 e^{2lt},$$

$$\frac{d}{dt}(e^{2lt}|\boldsymbol{u}|^2) \le \frac{b^2}{2(b-1)}(r+\sigma)^2 e^{2lt},$$

$$e^{2lt}|\boldsymbol{u}(\boldsymbol{t})|^2 \le \frac{b^2}{4l(b-1)}(r+\sigma)^2 e^{2lt} + C,$$

$$|\boldsymbol{u}(\boldsymbol{t})|^2 \le \frac{b^2}{4l(b-1)}(r+\sigma)^2 + Ce^{-2lt}$$

Taking constant $C = |\boldsymbol{u}(\boldsymbol{0})|^2 - \frac{b^2}{4l(b-1)}(r+\sigma)^2$, we have

$$|\boldsymbol{u}(\boldsymbol{t})|^2 \le \frac{b^2}{4l(b-1)}(r+\sigma)^2 + \left(|\boldsymbol{u}(\boldsymbol{0})|^2 - \frac{b^2}{4l(b-1)}(r+\sigma)^2\right)e^{-2lt},$$

$$|\boldsymbol{u}(\boldsymbol{t})|^2 \le |\boldsymbol{u}(\boldsymbol{0})|^2 e^{-2lt} + \frac{b^2}{4l(b-1)}(r+\sigma)^2(1-e^{-2lt}). \qquad (2.7)$$

Since $\sigma, b, r > 0$ and $l = \min(1, \sigma)$, inequality (2.7) becomes

$$|\boldsymbol{u}(\boldsymbol{t})| \le |\boldsymbol{u}(\boldsymbol{0})|e^{-lt} + \frac{b}{2\sqrt{l(b-1)}}(r+\sigma)(1-e^{-lt}) \qquad (2.8)$$

Taking the limit as $t \to \infty$, from both terms of the inequality (2.8), yields

$$\lim_{t\to\infty}\sup|\boldsymbol{u}(\boldsymbol{t})| \le \lim_{t\to\infty}\sup\left(|\boldsymbol{u}(\boldsymbol{0})|e^{-lt} + \frac{b}{2\sqrt{l(b-1)}}(r+\sigma)(1-e^{-lt})\right),$$

$$\lim_{t\to\infty}\sup|\boldsymbol{u}(\boldsymbol{t})| \le \lim_{t\to\infty}\sup(|\boldsymbol{u}(\boldsymbol{0})|e^{-lt}) + \lim_{t\to\infty}\sup\left(\frac{b}{2\sqrt{l(b-1)}}(r+\sigma)\right)$$

$$- \lim_{t\to\infty}\sup\left(\frac{b}{2\sqrt{l(b-1)}}(r+\sigma)e^{-lt}\right),$$

$$\limsup_{t\to\infty} |\boldsymbol{u}(t)| \le \frac{b}{2\sqrt{l(b-1)}}(r+\sigma) \tag{2.9}$$

Equation (2.9) is equivalent to

$$\limsup_{t\to\infty} |\boldsymbol{u}(t)| \le \rho_0, \text{ with } \rho_0 = \frac{b}{2\sqrt{l(b-1)}}(r+\sigma). \tag{2.10}$$

This proves that the solution of Lorenz system in (1.1) remains bounded as $t \to \infty$.

Next, we show the existence of the absorbing set. Consider equation (2.6) and Assumption A that

$$t_0(|\boldsymbol{u}(\boldsymbol{0})|) = \max\left(\frac{1}{2l}\ln\left(\frac{4l(b-1)|\boldsymbol{u}(\boldsymbol{0})|^2}{b^2(r+\sigma)^2}\right), 0\right).$$

Here we already assumed that

$$\rho_0^2 = \frac{b^2(r+\sigma)^2}{4l(b-1)}$$

so that $t_0(|\boldsymbol{u}(\boldsymbol{0})|)$ is equivalent to

$$t_0(|\boldsymbol{u}(\boldsymbol{0})|) = \max\left(\frac{1}{2l}\ln\left(\frac{|\boldsymbol{u}(\boldsymbol{0})|^2}{\rho_0^2}\right), 0\right)$$

Let us choose $t_0(|\boldsymbol{u}(\boldsymbol{0})|) = \frac{1}{2l}\ln\left(\frac{|\boldsymbol{u}(\boldsymbol{0})|^2}{\rho_0^2}\right)$ with $\rho_0^2 < |\boldsymbol{u}(\boldsymbol{0})|^2$ and R large enough so that it encloses the ball $\mathcal{B}_0 = B(0, \rho)$ which contains bounded sets $\mathcal{B}_i$ such that

$$\bigcup_{i=1}^{n} \mathcal{B}_i \subset \boldsymbol{B}(0, R) \tag{2.11}$$

with radius $\rho > \rho_0$, $\rho_0 = \frac{b}{2\sqrt{l(b-1)}}(r+\sigma)$.

Equation (2.11) proves that there exists an absorbing set $\mathcal{B}_0$ in $\mathbb{R}^3$.

Finally, we show that the ω-limit set is compact. From (2.9), we have that all sets in $\boldsymbol{B}(0, R)$ are bounded, thus the union of that sets still contains in $\boldsymbol{B}(0, R)$, such that

$$\bigcup_{i=1}^{n} \mathcal{B}_i \subset \boldsymbol{B}(0, R).$$

This allows us to define the semigroup of operators $\{T(t)\}$ on bounded sets $\mathcal{B}_i$ as follows:

$$\gamma^+(\mathcal{B}_n) = \bigcup_{t \geq t_0(\| \boldsymbol{u}(\mathbf{0}) \|)} T(t)\mathcal{B}_n$$

Then we take the closure of each orbit based on Definition 1.5. It follows that the ω-limit set is equivalent to

$$\left(\bigcup_{t \geq t_0(\| \boldsymbol{u}(\mathbf{0}) \|)} T(t)\mathcal{B}_n \right) \cup \left(\bigcup_{t \geq t_0(\| \boldsymbol{u}(\mathbf{0}) \|)} T(t)\mathcal{B}_n \right)' = cl_V \bigcup_{t \geq t_0(\| \boldsymbol{u}(\mathbf{0}) \|)} T(t)\mathcal{B}_n,$$

with $(\bigcup_{t \geq t_0(\| \boldsymbol{u}(\mathbf{0}) \|)} T(t)\mathcal{B}_i)'$ being the limit point of $\bigcup_{t \geq t_0(\| \boldsymbol{u}(\mathbf{0}) \|)} T(t)\mathcal{B}_i$.

Since $\mathcal{B}_i$ exist in the normed space V, from Corollary 1.1, we have that the set

$$cl_V \bigcup_{t \geq t_0(\| \boldsymbol{u}(\mathbf{0}) \|)}^{n} T(t)\mathcal{B}_i \tag{2.12}$$

is closed.

From Definition 1.5, we have

$$\omega(B) = \bigcap_{t_0(\| \boldsymbol{u}(\mathbf{0}) \|) > 0} cl_V \bigcup_{t \geq t_0(\| \boldsymbol{u}(\mathbf{0}) \|)} T(t)\mathcal{B}_i$$

with $i = 1, 2, \ldots, n$.

From (2.11) and (2.12), $\omega(B)$ is closed. Thus, $\omega(B)$ is bounded and hence conclude that $\omega(B)$ is compact.

To prove that $\omega(B)$ attracts $\mathcal{B}_i$, we take supremum $m_i \in T(t)\mathcal{B}_i$ and infimum $n \in \omega(B)$ such that

$$d(T(t)\mathcal{B}_i, \omega(B)) = \sup_{m_i \in T(t)\mathcal{B}_i} \inf_{n \in \omega(B)} d(m_i, n).$$

Then

$$d(T(t)\mathcal{B}_i, \omega(B)) \to 0 \text{ as } t \to +\infty.$$

Based on Definition 1.3 we prove that $\omega(B)$ is a compact attractor which attracts all

$$\bigcup_{i=1}^{n} \mathcal{B}_i \subset \boldsymbol{B}(0, R).$$

So the existence of global attractor in the Lorenz system is proved. ☐

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