Global Stability of Prey-Predator Model with CrowleyMartin type Functional Response and Stage Structure for Predator

by

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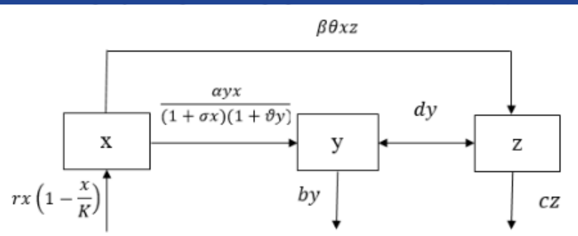

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The characteristic equation of the Jacobian matrix is determined by $\det(J - \lambda I) = 0$, i.e. $f(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$, where the coeficients of the polynomial come from the Jacobian matrix evaluated at the equilibrium point. According to the Routh-Hurwitz stability criterion, the equilibrium point with three variables is asymptotically stable if and only if $a_0 > 0$, $a_2 > 0$, and $a_2a_1 - a_0 > 0$ (Toaha & Azis, 2018). Next, the equilibrium point and its stability will be determined and analyzed to understand the sustainability of the predator-prey population and numerical simulations will be carried out to confirm the analytical result.

3.2 Equilibrium Point of the Model

Equilibrium point $TE(x, y, z)$ of equation (2) is obtained by analyzing and solving the equations $\frac{dx}{dt} = 0$, $\frac{dy}{dt} = 0$, and $\frac{dz}{dt} = 0$. Thus, from the equation (2), we have

$$rx - \frac{rx^2}{K} - \frac{\alpha yx}{(1+\sigma x)(1+\vartheta y)} - \beta zx = 0$$
$$\frac{\delta\alpha yx}{(1+\sigma x)(1+\vartheta y)} - (b + d)y = 0 \qquad (4)$$
$$\theta zx - cz + dy = 0.$$

There are four non-negative equilibrium points are obtained from equation (4). The equilibrium points are $TE_1(x, y, z) = (0, 0, 0)$, $TE_2(x, y, z) = (K, 0, 0)$, $TE_3(x, y, z) = \left(\frac{c}{\theta}, 0, \frac{r(K\theta - c)}{K\beta\theta}\right)$, and

$$TE_4(x, y, z) = \left(\omega, \frac{b\sigma\omega + d\sigma\omega - \alpha\omega + b + d}{(b + d)(\sigma\omega + 1)\vartheta}, \frac{d(b\sigma\omega + d\sigma - \alpha\omega + b + d)}{\vartheta(\sigma\omega + 1)(\theta\omega - c)(b + d)}\right).$$

The symbol ω in the equilibrium point above is the roots of the equation of $(br\sigma\theta\vartheta + dr\sigma\theta\vartheta)Z^4 + (-Kbr\sigma\theta\vartheta - Kdr\sigma\theta\vartheta - bcr\sigma\vartheta + br\theta\vartheta + dr\theta\vartheta)Z^3 + (Kbcr\sigma\vartheta + Kcdr\sigma\vartheta + Kb^2\sigma\theta + Kb\beta d\sigma - 2Kbd\sigma\theta - Kbr\theta\vartheta + K\beta d^2\sigma - Kd^2\sigma\theta - Kdr\theta\vartheta + K\alpha b\theta - K\alpha\beta d + K\alpha d\theta - bcr\vartheta - cdr\vartheta)Z^2 + (Kb^2c\sigma + 2Kbcd\sigma + Kbcr\vartheta + Kcd^2\sigma + Kcdr\vartheta - K\alpha bc - K\alpha cd - Kb^2\theta - Kb\beta d - 2Kbd\theta + K\beta d^2 - Kd^2\theta)Z + Kb^2c + 2Kbcd + Kcd^2$.

The equilibrium points $\boldsymbol{TE_1(x, y, z)}$, $\boldsymbol{TE_2(x, y, z)}$, and $\boldsymbol{TE_3(x, y, z)}$ are the equilibrium points with at least one of the compartment values is zero. These equilibrium points will not be considered because we just consider the interior equilibrium point. Because if the population compartment value is zero, then there is no growth and development of a population. The only one equilibrium point that will be analyzed is the equilibrium point $\boldsymbol{TE_4(x, y, z)}$ where the three point of components are positive. The equilibrium point $\boldsymbol{TE_4(x, y, z)}$ lies in the first octant when $\boldsymbol{\omega > 0}$, $\boldsymbol{a_1 + \omega^2 > 0}$ and $\boldsymbol{a_2 + \omega^2 > 0}$. The stability of equlibrium point $\boldsymbol{TE_4(x, y, z)}$ will be analyzed locally and globally.

3.3 Stability Analysis

Analysis of stability of the equilibrium point is carried out using the linearization method and determining the stability by taking into account the eigenvalues obtained from the Jacobian matrix which are evaluated at the equilibrium point. In equation (2) only the interior equilibrium point $\boldsymbol{TE_4(x,y,z)}$ will be analyzed for stability with a non-negative equilibrium point. By linearizing equation (2) using Jacobian matrix (3) we have

$$\frac{\partial f_1}{\partial x} = r - \frac{2rB}{K} - \frac{\alpha y}{\sigma\vartheta xy + \sigma x + \vartheta y + 1} + \frac{\alpha xy(\sigma y\vartheta + \sigma)}{(\sigma\vartheta xy + \sigma x + \vartheta y + 1)^2} - \beta z$$

$$\frac{\partial f_2}{\partial x} = \frac{\alpha y}{\sigma\vartheta xy + \sigma x + \vartheta y + 1} - \frac{\alpha xy(\sigma y\vartheta + \sigma)}{(\sigma\vartheta xy + \sigma x + \vartheta y + 1)^2}, \frac{\partial f_3}{\partial x} = \theta z$$

$$\frac{\partial f_1}{\partial y} = -\frac{\alpha x}{\sigma\vartheta xy + \sigma x + \vartheta y + 1} + \frac{\alpha xy(\sigma x\vartheta + \vartheta)}{(\sigma\vartheta xy + \sigma x + \vartheta y + 1)^2}$$

$$\frac{\partial f_2}{\partial y} = -b - d + \frac{\alpha x}{\sigma\vartheta xy + \sigma x + \vartheta y + 1} - \frac{\alpha xy(\sigma x\vartheta + \vartheta)}{(\sigma\vartheta xy + \sigma x + \vartheta y + 1)^2}$$

$$\frac{\partial f_3}{\partial y} = d, \frac{\partial f_1}{\partial z} = -\beta x, \frac{\partial f_2}{\partial z} = 0, \text{ and } \frac{\partial f_3}{\partial z} = \theta x - c.$$

By assuming $s = \frac{r}{K}$, $t = \frac{\alpha y}{\sigma\vartheta xy+\sigma x+\vartheta y+1}$, $u = \frac{\alpha xy(\sigma x\vartheta+\vartheta)}{(\sigma\vartheta xy+\sigma x+\vartheta y+1)^2}$, and $v = \frac{\alpha x}{\sigma\vartheta xy+\sigma x+\vartheta y+1}$, then the Jacobian matrix can be written as

$$\boldsymbol{J} = \begin{pmatrix} r - 2sx - t + u - \beta z & -v + u & -\beta x \\ t - u & -b - d + v - u & 0 \\ \theta z & d & \theta x - c \end{pmatrix}$$

Next, the characteristic equation of the Jacobian matrix will be determined by the equation $f(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$, as described in the following matrix

$$\lambda \boldsymbol{I} - \boldsymbol{J} = \begin{pmatrix} \lambda - r - 2sx - t + u - \beta z & -v + u & -\beta x \\ t - u & \lambda + b - d + v - u & 0 \\ \theta z & d & \lambda - \theta x - c \end{pmatrix}.$$

The determinant of matrix $(\lambda \boldsymbol{I} - \boldsymbol{J}) = \boldsymbol{0}$, $\det(\lambda \boldsymbol{I} - \boldsymbol{J}) = [(\lambda - r - 2sx - t + u - \beta z)(\lambda + b - d + v - u)(\lambda - \theta x - c) + (-\beta x)(t - u)(d)] - [(\theta z)(\lambda + b - d + v - u)(-\beta x) + (\lambda - \theta x - c)(t - u)(-v + u)]$.

Thus, the characteristic equation is obtained as follows

$$\lambda^3 + \beta z\lambda^2 + 2sx\lambda^2 - \theta x\lambda^2 + b\lambda^2 + c\lambda^2 + d\lambda^2 - r\lambda^2 + t\lambda^2 - v\lambda^2 - 2s\theta x^2\lambda + b\beta z\lambda + 2bsx\lambda - b\theta x\lambda + \beta cz\lambda + \beta dz\lambda + \beta uz\lambda - \beta vz\lambda + 2csx\lambda + 2dsx\lambda - d\theta x\lambda + r\theta x\lambda + 2sux\lambda - 2svx\lambda - t\theta x\lambda + \theta vx\lambda + bc\lambda - br\lambda + bt\lambda - bu\lambda + cd\lambda - cr\lambda + ct\lambda - cv\lambda - dr\lambda + dt\lambda - du\lambda - ru\lambda + rv\lambda - 2bs\theta x^2 - 2ds\theta x^2 - 2su\theta x^2 + 2sv\theta x^2 + b\beta cz + 2bcsx + br\theta x - bt\theta x + bu\theta x + \beta cdz + \beta cuz - \beta cvz + \beta dtx - \beta dux + 2cdsx + 2csux - 2csvx + dr\theta x - dt\theta x + du\theta x + r\theta ux - r\theta vx - bc + bct - bcu - cdr + cdt - cdu - cru + crv = 0.$$

The characteristic equation can be written in the form of the equation $f(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$, with

$$a_2 = \beta z + 2sx - \theta x + b + c + d - r + t - v,$$

$$a_1 = -2s\theta x^2 + b\beta z + 2bsx - b\theta x + \beta cz + \beta dz + \beta uz - \beta vz + 2csx + 2dsx - d\theta x + r\theta x + 2sux - 2svx - t\theta x + \theta vx + bc - br + bt - bu + cd - cr + ct - cv - dr + dt - du - ru + rv,$$

$$a_0 = -2bs\theta x^2 - 2ds\theta x^2 - 2su\theta x^2 + 2sv\theta x^2 + b\beta cz + 2bcsx + br\theta x - bt\theta x + bu\theta x + \beta cdz + \beta cuz - \beta cvz + \beta dtx - \beta dux + 2cdsx + 2csux - 2csvx + dr\theta x - dt\theta x + du\theta x + r\theta ux - r\theta vx - bc + bct - bcu - cdr + cdt - cdu - cru + crv.$$

The equilibrium point $\boldsymbol{TE_4(x,y,z)}$ becomes locally asymptotically stable when the requirements of Routh-Hurwitz stability criteria, namely $\boldsymbol{a_0 > 0}$, $\boldsymbol{a_2 > 0}$, and $\boldsymbol{a_2 a_1 - a_0 > 0}$ are satisfied. This means that if given a disturbance with the initial value populations, then the solution curve of the model will tend towards non-negative equilibrium point $\boldsymbol{TE_4(x,y,z)}$ as time goes on. These criteria will be demonstrated in the next numerical simulation.

3.4 Global Stability

To analyze the global stability of the equilibrium points in equation (4) that meet the local stability used the Lyapunov method with constructed the Lyapunov function.

Definition (Luenberger, 1979)

Given function $V: D \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ and $\bar{x} \in D$ is the equilibrium point of a non-linear system of differential equations. Function $V(x)$ is called a Lyapunov function if it satisfies all three statements, namely

- Function $V(x)$ is continuous and has a continuous first partial derivative at D or $\boldsymbol{V(x) \in C'(D)}$,

- Function $V(\boldsymbol{x}) > 0$ for $\boldsymbol{x} \in D$,
- Function $\dot{V}(\boldsymbol{x}) \leq 0$ for $\boldsymbol{x} \in D$.

From the three terms of the definition, if $\dot{V}(\boldsymbol{x}) \leq 0$ then the equilibrium point of system (2) is stable and if $\dot{V}(\boldsymbol{x}) > 0$ then the equilibrium point of system (2) is asymptotically stable. If $\dot{V}(\boldsymbol{x}) < 0$ and $(\boldsymbol{x})$ radially infinite, i.e

$$\|x\| \longrightarrow \infty \Rightarrow V(x) \longrightarrow \infty. \tag{5}$$

Then the equilibrium point of equation (2) is globally asymptotically stable. However, if the function $(\boldsymbol{x})$ does not satisfy equation (4) then the equilibrium point of equation (2) is locally asymptotically stable (Marquez, 2003).

Theorem (Marquez, 2003)

Given a non-linear system of differential equations $\frac{dx}{dt} = f(x)$ and Jacobian matrix of the form $J(x) = \left(\frac{\partial f}{\partial x}\right)$. If matrix $f(x) = J(x) + J^T(x)$ is a negative definite matrix for each $\boldsymbol{x} \in D$ $(\boldsymbol{0} \in D)$, then the equilibrium point is locally asymptotically stable and the Lyapunov function for the system is

$$V(x) = f^T(x) f(x). \tag{6}$$

In constructing the Lyapunov function, a Krasovskii method is needed which can fulfill all the requirements in forming the function. The Krasovskii method is a method invented by a Russian mathematician named Nikolay Krasovsky to create a method that can be used to construct Lyapunov functions through the presence of a positive definite Hermitian matrix. In order to determine the global stability analysis, the matrix $\boldsymbol{J}$ is transposed to obtain

$$J^T = \begin{pmatrix} r - 2sx - t + u - \beta z & t - u & \theta z \\ -v + u & -b - d + v - u & d \\ -\beta x & 0 & \theta x - c \end{pmatrix}.$$

In the Krasovsky method, the Lyapunov function $V(x)$ is constructed by forming $V(x) = f^T(x) P f(x)$, where P is a Hermitian matrix which is positive definite (significant symmetry) and $f(x)$ is a function of equation (2), that is

$$\boldsymbol{f} = \begin{pmatrix} rx - \frac{rx^2}{K} - \frac{\alpha yx}{1+\sigma x+\vartheta y+\sigma\vartheta xy} - \beta zx \\ \frac{\delta\alpha yx}{1+\sigma x+\vartheta y+\sigma\vartheta xy} - by - dy \\ \theta zx - cz + dy \end{pmatrix}.$$

Given any Hermitian matrix P, i.e

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}.$$

This matrix is a form of a real Hermitian matrix, because the elements are $p_{11}, \ldots, p_{33} \in \mathbb{R}$ and $p_{ij} = p_{ji}$ with $i, j = 1, 2, 3$ and $i \neq j$. Next, consider the function $V(\boldsymbol{x})$ which is differentiated with respect to t, so that

$$V(x) = f^T(x) P f(x)$$
$$\dot{V}(x) = \dot{f}^T(x) P f(x) + f^T(x) P \dot{f}(x)$$
$$\dot{V}(x) = \left(\frac{\partial f}{\partial x}\frac{dx}{dt}\right)^T P f(x) + f^T(x) P \left(\frac{\partial f}{\partial x}\frac{dx}{dt}\right)$$
$$\dot{V}(x) = \left(\frac{\partial f}{\partial x}\right)^T \left(\frac{dx}{dt}\right)^T P f + f^T P \left(\frac{\partial f}{\partial x}\right)\left(\frac{dx}{dt}\right)$$
$$\dot{V}(x) = J^T f^T(x) P f(x) + f^T(x) P J f(x)$$
$$\dot{V}(x) = f^T(x)(J^T P + PJ) f(x), \tag{7}$$

where $J = \frac{\partial f}{\partial x}$ is the Jacobian matrix. If matrix $J^T P + PJ$ is negative semidefinite then $\dot{V}(\boldsymbol{x})$ in equation (7) is also negative semidefinite. The implication statement is used to fulfill the conditions of $\dot{V}(\boldsymbol{x})$ negative semidefinite. In order to fulfill the conditions of $\dot{V}(\boldsymbol{x})$ negative semidefinite and $V(x)$ positive definition is formed matrix P by using the equation,

$$(J^T P + PJ) = -Q. \tag{8}$$

The matrix Q in equation (8) is a positive definite Hermitian matrix. To facilitate the formation of the Hermitian matrix P, Krasovsky method uses an identity matrix I which has the same size as the matrix P. The identity matrix I also satisfies a positive definite Hermitian matrix. Based on equation (7) then the matrix can be written as

$$\begin{pmatrix} r-2sx-t+u-\beta z & t-u & \theta z \\ -v+u & -b-d+v-u & d \\ -\beta x & 0 & \theta x - c \end{pmatrix}\begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix} + \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}\begin{pmatrix} r-2sx-t+u-\beta z & -v+u & -\beta x \\ t-u & -b-d+v-u & 0 \\ \theta z & d & \theta x - c \end{pmatrix} = -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Matrix P^* is the solution of equation (7), that is

$$P^* = \begin{pmatrix} p_{11}{}^* & p_{12}{}^* & p_{13}{}^* \\ p_{21}{}^* & p_{22}{}^* & p_{23}{}^* \\ p_{31}{}^* & p_{32}{}^* & p_{33}{}^* \end{pmatrix}.$$

If matrix P^* is positive definite then the Lyapunov function is obtained by substituting the Hermitian matrix P^* in equation $V(x) = f^T(x)P^*f(x)$. However, if there is no positive definite then recheck the matrix value P to get the matrix P^* positive definite. Because matrix P^* positive definite, then check whether the equilibrium point evaluated at the equation $V(x)$ fulfills equation (5).

3.5 Numerical Simulation

In order to determine the global stability of the prey-predator population dynamics, a numerical simulation was carried out by considering the parameter values in Table 2 below.

Table 2. Variables and Parameter Values

Variable/Parameters	Parameter Values	Dimension
K	120	Biomass
$\boldsymbol{r}$	0.7	$(\text{Year})^{-1}$
$\boldsymbol{\alpha}$	0.51	$(\text{Immature predator})^{-1}$ $(\text{year})^{-1}$
$\boldsymbol{\beta}$	0.255	$(\text{Mature predator})^{-1}$ $(\text{year})^{-1}$
$\boldsymbol{\sigma}$	0.08	$(\text{Day})^{-1}$
$\boldsymbol{\vartheta}$	0.05	$(\text{Immature predator})^{-1}$ $(\text{Day})^{-1}$
$\boldsymbol{\delta}$	0.1	$(\text{Prey})^{-1}$ $(\text{year})^{-1}$
$\boldsymbol{d}$	0.4	$(\text{Mature predator})^{-1}$ $(\text{Year})^{-1}$
$\boldsymbol{\theta}$	0.255	$(\text{Prey})^{-1}$ $(\text{year})^{-1}$
b	0.3	$(\text{immature predator})^{-1}$ $(\text{year})^{-1}$
c	0.4	$(\text{mature predator})^{-1}$ $(\text{year})^{-1}$

With these parameter values, four non-negative equilibrium points are obtained, namely $TE_1 = (0,\ 0,\ 0)$, $TE_2 = (120,\ 0,\ 0)$, $TE_3 = (1.56862,\ 0,\ 2.70921)$, and $TE_4 = (1.54525,\ 0.03933,\ 2.63987)$. The eigenvalues associates with the equilibrium point are obtained using Jacobian matrix

$$J = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

where

$$a_{11} = 0.7 - 0.1167x - \frac{0.51y}{0.004xy + 0.08x + 0.05y + 1} + \frac{0.51xy(0.004y + 0.08)}{(0.012xy + 0.01x + 1.2y + 1)^2} - 0.25z;$$

$$a_{12} = -\frac{0.51y}{0.004xy+0.08x+0.05y+1} + \frac{0.51xy(0.004y+0.08)}{(0.012xy+0.01x+1.2y+1)^2};\ a_{13} = -0.255x;$$

$$a_{21} = \frac{0.51y}{0.004xy + 0.08x + 0.05y + 1} - \frac{0.51xy(0.004y + 0.08)}{(0.012xy + 0.01x + 1.2y + 1)^2};\ a_{23} = 0;$$

$$a_{22} = -0.7 + \frac{0.51y}{0.004xy + 0.08x + 0.05y + 1} - \frac{0.51xy(0.004y + 0.08)}{(0.012xy + 0.01x + 1.2y + 1)^2};$$

$$a_{31} = 0.255z;\ a_{32} = 0.4;\ a_{33} = -0.4 + 0.255x.$$

By substituting the value of the equilibrium point $TE_4 = (1.54525, 0.03933, 2.63987)$ in Jacobian matrix $\boldsymbol{J}$, we get

$$J = \begin{pmatrix} -0.0007261688 & -0.697190909 & -0.39270 \\ 0.01210950212 & -0.002840909136 & 0 \\ 0.67065 & 0.4 & -0.00730 \end{pmatrix}.$$

From the Jacobian matrix, we have the characteristics $f(\lambda) = \lambda^3 + 0.01086707794\lambda^2 + 0.2718346071\lambda + 0.002711997993$ with three eigenvalues $\lambda_1 = -0.00044505 + 0.52136889\,\mathrm{I}$, $\lambda_2 = -0.00044505 - 0.52136889\,\mathrm{I}$, and $\lambda_3 = -0.00997697321$[illegible]$8181$. From the three eigenvalues we conclude that the equilibrium point $TE_4 = (1.54525, 0.03933, 2.63987)$ is locally asymptotically stable and also satisfies the Routh-Hurwitz stability criteria with the following conditions $a_0 > 0$, $a_2 > 0$, and $a_2 a_1 - a_0 > 0$, where $a_2 = 0.01086707794$, $a_1 = 0.2718346071$, and $a_0 = 0.002711997993$.

Now we will analyze whether the equilibrium point is globally asymptotically stable. The analysis uses the Lyapunov method, namely by constructing the Lyapunov function. The global stability of equilibrium point via Lyapunov function will be constructed using the Krasovskii method. The construction of the Lyapunov function that will be produced by taking the form

$$V(x) = f^T(x) P^* f(x),$$

where

$$f(x) = \begin{pmatrix} 0.7 - 0.005833x - \frac{0.51y}{0.004xy + 0.08x + 0.05y + 1} - 0.255z \\ -0.7y + \frac{0.51y}{0.004xy + 0.08x + 0.05y + 1} \\ -0.4z + 0.255zx + 0.4y \end{pmatrix}.$$

and

$$f^T(x) = \left(\left(0.7 - 0.005833x - \frac{0.51y}{0.004xy + 0.08x + 0.05y + 1} - 0.255z\right) \left(-0.7y + \frac{0.51y}{0.004xy + 0.08x + 0.05y + 1}\right) (-0.4z + 0.255zx + 0.4y) \right).$$

In order to fulfill $V^*(x)$ to be negative semi definite and $V(x)$ to be positive definite, take matrix P^* by using equation (7). The transpose form of the Jacobian matrix is

$$J^T = \begin{pmatrix} -0.0007261688 & 0.01210950212 & 0.67065 \\ -0.697190909 & -0.002840909136 & 0.4 \\ -0.39270 & 0 & -0.00730 \end{pmatrix}.$$

Because of J and J^T have been obtained, then equation (7) can be worked out to get all the elements of the matrix P^* which is as follows

$$\begin{pmatrix} -0.0007261688 & 0.01210950212 & 0.67065 \\ -0.697190909 & -0.002840909136 & 0.4 \\ -0.39270 & 0 & -0.00730 \end{pmatrix} \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix} + \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix} \begin{pmatrix} -0.0007261688 & -0.697190909 & -0.39270 \\ 0.01210950212 & -0.002840909136 & 0 \\ 0.67065 & 0.4 & -0.00730 \end{pmatrix} = -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

From which we have

$$\begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix} = -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where

$q_{11} = -0.0014523376p_{11} + 0.01210950212p_{21} + 0.67065\, p_{31} + 0.01210950212\, p_{12} + 0.67065\, p_{13}$,

$q_{12} = -0.003567077936\, p_{12} + 0.01210950212\, p_{22} + 0.67065\, p_{32} - 0.6971590909\, p_{11} + 0.4\, p_{13}$,

$q_{13} = -0.0080261688\, p_{13} + 0.01210950212\, p_{23} + 0.67065\, p_{33} - 0.39270\, p_{11}$,

$q_{21} = -0.6971590909\, p_{11} - 0.003567077936\, p_{21} + 0.4\, p_{31} + 0.01210950212\, p_{22} + 0.67065\, p_{23}$,

$q_{22} = -0.6971590909\, p_{12} - 0.005681818272\, p_{22} + 0.4\, p_{32} - 0.6971590909\, p_{21} + 0.4\, p_{23}$,

$q_{23} = -0.6971590909\, p_{13} - 0.01014090914\, p_{23} + 0.4\, p_{33} - 0.39270\, p_{21}$,

q_{31} =-0.39270 p_{11}-0.0080261688 p_{31}+0.01210950212 p_{32}+0.67065 p_{33},

q_{32} =-0.39270 p_{12}-0.01014090914 p_{32}-0.6971590909 p_{31}+0.4 p_{33},

q_{33} =-0.39270 p_{13}-0.01460 p_{33}-0.39270p_{31}.

To determine the elements of the matrix P^*, Cramer's method is used to solve a system of linear equations by converting it into a matrix form. The Cramer method uses the determinant of a matrix and another matrix obtained by replacing one of the columns with a vector consisting of the numbers to the right of the equation, namely the elements of the Hermitian matrix. The form of a matrix of order 9x9 as shown below,

$$\begin{pmatrix} -0.00145 & 0.01210 & \dots & 0 \\ -0.69715 & -0.0035 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -0.0146 \end{pmatrix} \begin{pmatrix} p_{11} \\ p_{12} \\ \vdots \\ p_{33} \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ \vdots \\ -1 \end{pmatrix}$$

Thus, based on Cramer's method, the elements of the matrix P^*,

$$P^* = \begin{pmatrix} p_{11}{}^* & p_{12}{}^* & p_{13}{}^* \\ p_{21}{}^* & p_{22}{}^* & p_{23}{}^* \\ p_{31}{}^* & p_{32}{}^* & p_{33}{}^* \end{pmatrix} = \begin{pmatrix} 1491.420937 & 848.6093281 & -14.45353822 \\ 848.6093281 & 3407.097322 & 1501.987498 \\ -14.45353822 & 1501.987498 & 846.0101797 \end{pmatrix}.$$

Next, matrix P^* is determined whether is negative definite or not by taking into account the eigenvalues. The eigenvalues obtained are $\lambda_1 = 48.1213350185760$, $\lambda_2 = 1386.32710869567$, and $\lambda_3 = 4310.07999498577$. Since all the eigenvalues are positive, then the matrix P^* is positive definite, so the Lyapunov function is obtained as follows,

$$V(x) = f^T(x)P^*f(x).$$

This can be written as

$$V(x) = \left(1043.9946x - 8.6999x^2 - \frac{327.8339yx}{0.004xy+0.08x+0.05y+1} - 383.9979zx - 599.809y + 5.7814z\right)$$
$$\left(0.7x - 0.0058x^2 - \frac{0.51yx}{0.004xy+0.08x+0.05y+1} - 0.255zx\right) + (594.0265x - 4.9502x^2 -$$
$$\frac{1304.8288yx}{0.004xy+0.08x+0.05y+1} - 166.6114zx - 1784.1731y - 600.7949z)\left(-0.7x + \frac{0.51yx}{0.004xy+0.08x+0.05y+1}\right)$$
$$+(-10.1174x + 0.0843x^2 - \frac{773.3849yx}{0.004xy+0.08x+0.05y+1} + 219.4182zx - 712.9871y -$$
$$338.4040)(0.255zx + 0.4y - 0.4z).$$

By substituting the value of the equilibrium point $TE_4 = (1.54525, 0.03933, 2.63987)$ then we get the value $V(x) = 0.2081225355$. Since the value of the equation $V(x) > 0$ then it can be concluded that the equilibrium point TE_4 globally asymptotically stable. If the prey population, immature predator, and mature predator were initially around the interior equilibrium point, then the three populations will tend toward the equilibrium point $TE_4 = (1.54525, 0.03933, 2.63987)$. This means that the three populations will not be extinct for a long period of time.

4. Conclusion

The prey-predator model with Crowly-Martin type functional response and stages structure for predator population has only one non-negative equilibrium point $\boldsymbol{TE_4(x, y, z)} = \left(\boldsymbol{\omega}, \frac{\boldsymbol{b\sigma\omega + d\sigma\omega - \alpha\omega + b + d}}{\boldsymbol{(b+d)(\sigma\omega+1)\vartheta}}, \frac{\boldsymbol{d(b\sigma\omega + d\sigma - \alpha\omega + b + d)}}{\boldsymbol{\vartheta(\sigma\omega+1)(\theta\omega - c)(b+d)}}\right)$. This equilibrium point becomes the only interior equilibrium point and it is locally and globally asymptotically stable when a certain conditions are fulfilled. This means that even if there is a tight interaction or predation on prey over a long period of time, the prey population will still be sustainable, stable, and sustainably maintained. This global stability provides an interpretation that the populations in the ecosystem is under control for a long period of time. As suggestions for further research, the model can be developed by considering some assumptions or consider the time delay and harvesting in the mechanism of growth of each prey and predator population.

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