

# Discrete Double-Sided Quaternionic Fourier Transform and Application

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# Discrete Double-Sided Quaternionic Fourier Transform and Application

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**Abstract.** In the present article, we first introduce the definition of the discrete double-sided quaternionic Fourier transform (DQFT) and obtain its inverse. We give examples how to compute the DQFT. We derive a discrete version of the duality property of the DQFT. We finally present an application of the DQFT to study the two-dimensional discrete linear time-varying systems.

## 1. Introduction

In the past few years, research on quaternionic Fourier transform has developed rapidly (see, e.g. [1, 2, 3, 4, 5, 6]). Basically, there are two kinds of the quaternionic Fourier transforms, namely the discrete quaternionic Fourier transform and continuous quaternionic Fourier transform [7, 8, 9, 10, 11, 12]. The discrete double-sided quaternionic Fourier transform is a generalization of the discrete Fourier transformation [13] in the setting of quaternion algebra. Since in some applications, we are always dealing with the discrete quaternionic Fourier transform [14, 15, 16], it is important to study the properties and its relation to the other general transform. Because the DQFT is a general form of the classical discrete Fourier transform (DFT), then several results of the DFT can be transferred to the DQFT domain.

In this article, we first provide the definition and inverse of the discrete double-sided quaternionic Fourier transform (DQFT). We give examples to compute the DQFT of an image and show that the original image can be obtained from its DQFT. We investigate in detail a discrete version of the duality property of the DQFT. We finally show that the DQFT can be used to studying the two-dimensional (2-D) discrete linear time-varying systems.

The remnant of the present work has been structured as follows. In Section 2 we recall basic results of the quaternions which will be needed later on. Section 3 provide definition and inverse of the DQFT and obtain theorem. In this section we also presents examples to compute the DQFT. Section 4 studies how to apply the DQFT for studying the 2-D discrete linear time-varying systems. In Section 5 we conclude the paper.

## 2. Notation

The quaternion algebra over real number  $\mathbb{R}$  is defined by

$$\mathbb{H} = \{s = s_a + \mathbf{i}s_b + \mathbf{j}s_c + \mathbf{k}s_d \mid s_a, s_b, s_c, s_d \in \mathbb{R}\}, \quad (1)$$

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9 which is an 7 associative non-commutative four-dimensional algebra. The basic elements  $\{i, j, k\}$  satisfy the rules:

$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j, \quad i^2 = j^2 = k^2 = ijk = -1. \quad (2)$$

15 As in case of complex number, the conjugate of a quaternion  $z$  is defined by

$$\bar{s} = s_a - i s_b - j s_c - k s_d, \quad (3)$$

and it satisfies

$$\overline{s z} = \bar{z} \bar{s}. \quad (4)$$

From (3), we easily get

$$|s| = \sqrt{s \bar{s}} = \sqrt{s_a^2 + s_b^2 + s_c^2 + s_d^2}. \quad (5)$$

It is not difficult to see that

$$|sz| = |s||z|, \quad \forall s, z \in \mathbb{H}. \quad (6)$$

In view of (3) and (5), we can get the inverse of  $s \in \mathbb{H} \setminus \{0\}$  as

$$s^{-1} = \frac{\bar{s}}{|s|^2}. \quad (7)$$

5 For future use, we introduce a finite sequence of the quaternion numbers by

$$\{g(t, w), 0 \leq t \leq T, 0 \leq w \leq W\}. \quad (8)$$

### 3. Discrete QFT and its Inverse

In analogy with the discrete Fourier transformation, the discrete quaternionic Fourier transform (DQFT) can be defined as

**Definition 3.1.** Given  $g(t, w) \in \mathbb{H}^{T \times W}$  of the two-dimensional discrete quaternion function. The DQFT of  $g(t, w)$  is given by

$$Q_g(u, v) = \sum_{t=0}^{T-1} \sum_{w=0}^{W-1} e^{-\frac{2\pi i u t}{T}} g(t, w) e^{-\frac{2\pi j v w}{W}}. \quad (9)$$

Observe now that according to equation (1), the above identity can be expressed as

$$\begin{aligned} Q_g(u, v) &= \sum_{t=0}^{T-1} \sum_{w=0}^{W-1} e^{-\frac{2\pi i u t}{T}} (g_a(t, w) + i g_b(t, w) + j g_c(t, w) + k g_d(t, w)) e^{-\frac{2\pi j v w}{W}} \\ &= \sum_{t=0}^{T-1} \sum_{w=0}^{W-1} e^{-\frac{2\pi i u t}{T}} (g_a(t, w) + i g_b(t, w) + g_c(t, w) j + i g_d(t, w)) j e^{-\frac{2\pi j v w}{W}} \\ &= \sum_{t=0}^{T-1} \sum_{w=0}^{W-1} e^{-\frac{2\pi i u t}{T}} g_a(t, w) e^{-\frac{2\pi j v w}{W}} + i \sum_{t=0}^{T-1} \sum_{w=0}^{W-1} e^{-\frac{2\pi i u t}{T}} g_b(t, w) e^{-\frac{2\pi j v w}{W}} \\ &\quad + \sum_{t=0}^{T-1} \sum_{w=0}^{W-1} e^{-\frac{2\pi i u t}{T}} g_c(t, w) e^{-\frac{2\pi j v w}{W}} j + i \sum_{t=0}^{T-1} \sum_{w=0}^{W-1} e^{-\frac{2\pi i u t}{T}} g_d(t, w) e^{-\frac{2\pi j v w}{W}} j. \end{aligned} \quad (10)$$

For the two-dimensional discrete real image  $g$ , equation (10) takes the form

$$Q_g(u, v) = \sum_{t=0}^{T-1} \sum_{w=0}^{W-1} e^{-\frac{2\pi i ut}{T}} g_a(t, w) e^{-\frac{2\pi j vw}{W}}. \quad (11)$$

For an illustration, let us consider how compute the DQFT of a simple image of size  $2 \times 2$  pixels.

**Example 3.1.** Given an image  $g = \begin{bmatrix} 26 & 20 \\ 10 & 5 \end{bmatrix}$ . Then the DQFT of  $g$  is given by

$$Q_g = \begin{bmatrix} 61 & 11 \\ 31 & 1 \end{bmatrix}. \quad (12)$$

In view of equation (11) we can easily obtain

$$\begin{aligned} Q_g(0, 0) &= 26 + 20 + 10 + 5 = 61 \\ Q_g(0, 1) &= g(0, 0) + g(0, 1) e^{-\frac{2\pi j}{2}} + g(1, 0) + g(1, 1) e^{-\frac{2\pi j 1}{2}} \\ &= 26 - 20 + 10 - 5 = 11 \\ Q_g(1, 0) &= g(0, 0) + g(0, 1) + e^{-\frac{2\pi i}{2}} g(1, 0) + e^{-\frac{2\pi i}{2}} g(1, 1) \\ &= 26 + 20 - 10 - 5 = 31 \\ Q_g(1, 1) &= g(0, 0) + g(0, 1) e^{-\frac{2\pi j}{2}} + e^{-\frac{2\pi i}{2}} g(1, 0) + e^{-\frac{2\pi i}{2}} g(1, 1) e^{-\frac{2\pi j}{2}} \\ &= 26 - 20 - 10 + 5 = 1. \end{aligned} \quad (13)$$

**Definition 3.2.** The inverse two-dimensional discrete quaternion function (IDQFT) is defined by

$$(Q_g^{-1}g)(t, w) = \frac{1}{TW} \sum_{t=0}^{T-1} \sum_{w=0}^{W-1} e^{\frac{2\pi i ut}{T}} g(u, v) e^{\frac{2\pi j vw}{W}}. \quad (14)$$

**Theorem 3.1.** For a discrete quaternion function  $g$  we have

$$Q_g^{-1}(Q_g(u, v))(t, w) = g(t, w). \quad (15)$$

*Proof.* By using Theorem 3.1 we immediately obtain

$$\begin{aligned} Q^{-1}(Q(u, v))(t, w) &= \frac{1}{TW} \sum_{t=0}^{T-1} \sum_{w=0}^{W-1} e^{\frac{2\pi i ut}{T}} Q(u, v) e^{\frac{2\pi j vw}{W}} \\ &= \frac{1}{TW} \sum_{t=0}^{T-1} \sum_{w=0}^{W-1} \sum_{l=0}^{T-1} \sum_{r=0}^{W-1} e^{\frac{2\pi i ut}{T}} e^{-\frac{2\pi i lt}{T}} g(r, l) e^{-\frac{2\pi j rw}{W}} e^{\frac{2\pi j vw}{W}} \\ &= \frac{1}{TW} \sum_{t=0}^{T-1} \sum_{w=0}^{W-1} \sum_{l=0}^{T-1} \sum_{r=0}^{W-1} e^{\frac{2\pi i t(u-l)}{T}} g(r, l) e^{\frac{2\pi j (v-r)w}{W}}. \end{aligned} \quad (16)$$

It is obvious for  $u = l$  and  $v = r$  we obtain  $e^{\frac{2\pi i t(u-l)}{T}} = 1$  and  $e^{\frac{2\pi j (v-r)w}{W}} = 1$ . This facts give

$$\sum_{t=0}^{T-1} \sum_{w=0}^{W-1} \sum_{l=0}^{T-1} \sum_{r=0}^{W-1} e^{\frac{2\pi i t(u-l)}{T}} e^{\frac{2\pi j (v-r)w}{W}} = TW \quad (17)$$

However, for  $u \neq l$  and  $v \neq r$  we obtain

$$\sum_{t=0}^{T-1} \sum_{w=0}^{W-1} \sum_{l=0}^{T-1} \sum_{r=0}^{W-1} e^{\frac{2\pi i t(u-l)}{T}} e^{\frac{2\pi j(v-r)w}{W}} = 0. \quad (18)$$

In accordance with (17) and (18) we finish <sup>5</sup> the proof of the theorem.  $\square$

The following examples describes that the original image can easily be obtained by taking inverse of the DQFT.

**Example 3.2.** Let be  $Q_g = \begin{bmatrix} 61 & 11 \\ 31 & 1 \end{bmatrix}$  be the DQFT of  $g$ . By using the IDQFT we obtain the original image as

$$g = \begin{bmatrix} 26 & 20 \\ 10 & 5 \end{bmatrix}. \quad (19)$$

This can be obtained from

$$\begin{aligned} g(0,0) &= \frac{1}{4}(61 + 11 + 31 + 1) = 61 \\ g(0,1) &= \frac{1}{4}(Q_g(0,0) + Q_g(0,1)e^{\frac{2\pi j}{2}} + Q_g(1,0) + Q_g(1,1)e^{\frac{2\pi j1}{2}}) \\ &= \frac{1}{4}(61 - 11 + 31 - 1) = 20 \\ g(1,0) &= \frac{1}{4}(Q_g(0,0) + Q_g(0,1) + e^{\frac{2\pi i}{2}}Q_g(1,0) + e^{\frac{2\pi i}{2}}Q_g(1,1)) \\ &= \frac{1}{4}(61 + 11 - 31 - 1) = 10 \\ g(1,1) &= \frac{1}{4}(Q_g(0,0) + Q_g(0,1)e^{\frac{2\pi j}{2}} + e^{\frac{2\pi i}{2}}Q_g(1,0) + e^{-\frac{2\pi i}{2}}Q_g(1,1)e^{\frac{2\pi j}{2}}) \\ &= \frac{1}{4}(61 - 11 - 31 + 1) = 5. \end{aligned} \quad (20)$$

Now we are ready to state an important property of the DQFT (compare to [17]).

**Theorem 3.2** (DQFT duality). Let  $Q_g(u, v)$  be a DQF of the discrete quaternion function  $g$ . Then we have

$$Q_g(Q_g(u, v))(l, r) = g(-l, -r). \quad (21)$$

*Proof.* It directly follows from (9) that

$$\begin{aligned} Q_g(Q_g(u, v))(l, r) &= \sum_{u=0}^{T-1} \sum_{w=0}^{W-1} e^{-\frac{2\pi i l u}{T}} Q_g(u, v) e^{-\frac{2\pi j r v}{W}} \\ &= \sum_{u=0}^{T-1} \sum_{v=0}^{W-1} \sum_{t=0}^{T-1} \sum_{w=0}^{W-1} e^{-\frac{2\pi i l u}{T}} e^{-\frac{2\pi i u t}{T}} g(t, w) e^{-\frac{2\pi j r v}{W}} e^{-\frac{2\pi j v w}{W}} \\ &= \sum_{u=0}^{T-1} \sum_{w=0}^{W-1} \sum_{t=0}^{T-1} \sum_{w=0}^{W-1} e^{-\frac{2\pi i u(l+t)}{T}} g(t, w) e^{-\frac{2\pi j v(r+w)}{W}}. \end{aligned} \quad (22)$$

Notice first that if  $l + t \neq 0$  and  $r + w \neq 0$ , then it holds

$$\sum_{t=0}^{T-1} \sum_{w=0}^{W-1} e^{-\frac{2\pi i u(l+t)}{T}} e^{-\frac{2\pi j v(r+w)}{W}} = 0. \quad (23)$$

This implies that  $Q_g(Q_g(u, v))(l, r) = 0$ . For  $l + t = 0$  and  $r + w = 0$  equation (22) becomes

$$Q_g(Q_g(u, v))(l, r) = g(-l, -r). \quad (24)$$

This is the desired result.  $\square$

#### 4. Application of DQFT

In what follows, we shall present a simple application of the DQFT for studying discrete version of linear time-varying systems. Let us now start by introducing the definition below.

**Definition 4.1.** Given a two-dimensional discrete linear TV system. Let  $h_1(\cdot, \cdot)$  and  $h_2(\cdot, \cdot)$  be the quaternion impulse response of the filters. We define the output  $r(\cdot, \cdot)$  of the system to the input  $f(\cdot, \cdot)$  as

$$r(t, w) = \sum_{u=-\infty}^{\infty} \sum_{v=-\infty}^{\infty} h_1(t, t-u) f(u, v) h_2(w, w-v). \quad (25)$$

Next, we define the quaternion transfer function of the linear time-varying systems filter  $h$  as

$$R(t, w, \omega_1, \omega_2) = \sum_{t'=-\infty}^{\infty} \sum_{w'=-\infty}^{\infty} e^{-2\pi i t' \omega_1} h(t, w, t', w') e^{-2\pi j w' \omega_2}. \quad (26)$$

The following simple theorem relates the DQFT to the output of a discrete linear TV band-pass filter.

**Theorem 4.1.** Given a 2-D discrete linear time-varying system with the impulse response  $h$  determined by

$$h_1(t, t') h_2(t, w') = e^{-\frac{2\pi i t(t-t')}{T}} e^{-\frac{2\pi j w(w-w')}{W}}, \quad \text{for } 0 \leq t \leq T-1, 0 \leq w \leq W-1. \quad (27)$$

Suppose that the input to this system is the quaternion signal  $g(u, v)$ , then its output  $r(\cdot, \cdot)$  is the DQFT of  $g(u, v)$ .

*Proof.* Applying (25) results in

$$\begin{aligned} r(t, w) &= \sum_{u=-\infty}^{\infty} \sum_{v=-\infty}^{\infty} h_1(t, t-u) g(u, v) h_2(w, w-v) \\ &= \sum_{t=0}^{T-1} \sum_{w=0}^{W-1} e^{-\frac{2\pi i t(t-(t-u))}{T}} g(u, v) e^{-\frac{2\pi j w(w-(w-v))}{W}} \\ &= \sum_{t=0}^{T-1} \sum_{w=0}^{W-1} e^{-\frac{2\pi i u t}{T}} g(u, v) e^{-\frac{2\pi j v w}{W}}, \end{aligned} \quad (28)$$

which achieves the proof by the theorem.  $\square$

Now when impulse response  $h$  is described by

$$h_1(t, t')h_2(n, w') = \frac{1}{TW} e^{\frac{2\pi i t(t-t')}{T}} e^{\frac{2\pi j w(w-w')}{W}}, \quad (29)$$

thus from equation (25) we obtain

$$\begin{aligned} r_1(t, w) &= \sum_{u=-\infty}^{\infty} \sum_{v=-\infty}^{\infty} h_1(t, t-u) Q_g(u, v) h(w, w-v) \\ &= \frac{1}{TW} \sum_{t=0}^{T-1} \sum_{w=0}^{W-1} e^{\frac{2\pi i t(t-(t-u))}{T}} Q_g(u, v) e^{\frac{2\pi i w(w-(w-v))}{W}} \\ &= \frac{1}{TW} \sum_{t=0}^{T-1} \sum_{w=0}^{W-1} e^{\frac{2\pi i u t}{T}} Q_g(u, v) e^{\frac{2\pi j v w}{W}}. \end{aligned} \quad (30)$$

Here the input to the system is quaternion function  $Q_g(u, v)$ .

From (27), (29) we conclude the choice of the quaternionic impulse response gives output characteristics of the discrete linear time-varying systems.

## 5. Conclusion

In this work we have proposed the discrete double-sided quaternionic Fourier transform. We then have built a discrete version of the duality property of the DQFT. We finally discussed how to apply the DQFT for studying the 2-D discrete linear time-varying systems.

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