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by

AIMS Mathematics
http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(2): 2810–2819.
DOI: 10.3934/math.2022155
Received: 05 October 2021
Accepted: 14 November 2021
Published: 19 November 2021

Research article

Angle in the space of p-summable sequences

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Abstract: The aim of this paper is to investigate completness of A that equipped with usual norm on p-summable sequences space where A is subspace in p-summable sequences space and $1 \leq p < \infty$. We also introduce a new inner product on A and prove completness of A using a new norm that corresponds this new inner product. Moreover, we discuss the angle between two vectors and two subspaces in A. In particular, we discuss the angle between 1-dimensional subspace and $(s-1)$-dimensional subspace where $s \geq 2$ of A.

Keywords: completness; vectors; subspaces; ℓ^p spaces; angles
Mathematics Subject Classification: 15A03, 46B20, 46A45, 51N15

1. Introduction

In an inner product space, we can calculate angles between two subspaces. Let $(X, \langle \cdot, \cdot \rangle)$ be a real inner product space. If $U = \text{span}\{u\}$ is a 1-dimensional subspace and $V = \text{span}\{v_1, \cdots, v_t\}$ is a t-dimensional subspace of X, then the angle between subspaces U and V is defined by θ with $0 \leq \theta \leq \frac{\pi}{2}$ and $\cos^2\theta = \frac{\langle u, u_V\rangle^2}{\|u\|^2 \|u_V\|^2}$. In formula, u_V denotes the (orthogonal) projection of u on V and $\|\cdot\| = \langle \cdot, \cdot \rangle^2$. Gunawan et al. [3] show that the value of $\cos\theta$ is equal to the ratio between the length of the projection of u on V and the length of u $\left(\cos^2\theta = \frac{\|u_V\|^2}{\|u\|^2}\right)$. Likewise, if $U = \text{span}\{u, w_2, \cdots, w_s\}$ and $V = \text{span}\{v, w_2, \cdots, w_s\}$ are s-dimensional subspaces of X that intersects on $(s-1)$-dimensional subspace $W = \text{span}\{w_2, \cdots, w_s\}$ with $s \geq 2$ then the angle between U dan V is defined by θ with $0 \leq \theta \leq \frac{\pi}{2}$ and $\cos^2\theta = \frac{\langle u_W^\perp, v_W^\perp\rangle^2}{\|u_W^\perp\|^2 \|v_W^\perp\|^2}$ with $u_W^\perp$, $v_W^\perp$ are the orthogonal complement of u and v, respectively, on W. Gunawan et al. [3] show that the value of $\cos\theta$ is equal to the ratio between the volume of the s-dimensional parallelepiped spanned by the projection of $u, w_2, \cdots, w_s$ on V and the volume of the s-dimensional parallelepiped spanned by $u, w_2, \cdots, w_s$.

Now suppose $(X, \|\cdot\|)$ is a normed space. As it is known, not all normed spaces are inner product spaces. For instance, the space ℓ^p for $1 \le p < \infty$ with norm $\|x\|_p = \left[\sum_{n=1}^{\infty} |x_n|^p\right]^{\frac{1}{p}}$ is not an inner product space except for $p = 2$ (see [6]). Konca et al. [5] define a (weighted) inner product $\langle \cdot, \cdot \rangle_v$ on ℓ^p where $2 < p < \infty$, but $(\ell^p, \langle \cdot, \cdot \rangle_v)$ is not Banach space. They also find that the inner product is actually defined on a larger space namely $\ell_{2,v}$ that contains ℓ^p and $(\ell_{2,v}, \langle \cdot, \cdot \rangle_v)$ is Banach space. More recent works may be found in [4, 8, 12]

Now suppose $(A, \|\cdot\|)$ is a normed space where A is subspace of X . Two questions arise: What are A complete? Can we define a norm on A which satisfies the parallelogram law? The reason why we are interested in the parallelogram law is because we eventually wish to define an inner product, so that we can define angle between two subspaces and many other notions on this space. The concept of the angle between two vectors and two subspaces in normed space has been studied intensively, see [1, 2, 7, 9–11].

Let $(\ell^p, \|\cdot\|_p)$ be a normed space and A is subspace in ℓ^p for $1 \le p < \infty$. In this paper, we discuss completness of A that equipped with usual norm on ℓ^p. We also introduce a new inner product $\langle \cdot, \cdot \rangle_{b,2}$ on A such that $\left(A, \|\cdot\|_{b,2}\right)$ is complete where $\|x\|_{b,2} = \left(\langle x, x \rangle_{b,2}\right)^{\frac{1}{2}}$ denotes the induced norm in A. Motivated by this fact, we shall discuss the angle between two subspaces in A.

2. Main results

2.1. The completness of subspaces on ℓ^p

Let $\{x_1, \ldots, x_n\}$ be linearly independent set on ℓ^p for $1 \le p < \infty$. Define $A = \text{span}\{x_1, \ldots, x_n\}$. Then we observe that $\left(A, \|\cdot\|_p\right)$ is a subspace of $\left(\ell^p, \|\cdot\|_p\right)$. In [6], we know that ℓ^p is Banach space. Here, A will be proved as Banach space. Before this result, we have the following lemma.

Lemma 2.1. *Let* $C = \sum_{i=1}^{n} a_i$ *where* $a_1, \ldots, a_n \in R$.

(a) If $sign(a_j) = sign(C)$ *for* $j = 1, \ldots, n$ *then the equality*

$$\left|ta_j\right| + \left|(1-t)\,a_j + \sum_{\substack{i=1 \\ i \neq j}}^{n} a_i\right| = |C|$$

holds for some $t \in (0, 1)$.

(b) If $sign(a_j) = -sign(C)$ *for* $j = 1, \ldots, n$ *then the equality*

$$\left|ta_j\right| + \left|(t+1)\,a_j + \sum_{\substack{i=1 \\ i \neq j}}^{n} a_i\right| = |C|$$

holds for some $t \in (0, 1)$.

Proof. (a) Without loss of generality, writing $j = 1$. Suppose that $\text{sign}(a_1) = \text{sign}(C)$. Choose $t \in (0, 1)$ such that

$$D = (1-t)\,a_1 + \sum_{i=2}^{n} a_i.$$

and $\text{sign}(D) = \text{sign}(C)$. We can write $C = ta_1 + D$. Hence, $|C| = |ta_1| + |D|$.
(b) Without loss of generality, writing $j = 1$ and $\text{sign}(C) = 1$. Suppose that $\text{sign}(a_1) = -\text{sign}(C)$. Choose $t \in (0,1)$ such that

$$\begin{aligned} C &= -ta_1 + (1+t)\,a_1 + \sum_{i=2}^{n} a_i \\ &= -ta_1 + E \end{aligned}$$

and $\text{sign}(E) = \text{sign}(C)$. We observe that C, $-ta_1$, E are positive real. Hence, we can write $|C| = |ta_1| + |E|$. □

Using Lemma 2.1, We shall now show that $\left(A, \|\cdot\|_p\right)$ is Banach space.

Theorem 2.2. *Let $\{x_1, \ldots, x_n\}$ be linearly independent set on ℓ^p where $1 \leq p < \infty$. If $A = span\{x_1, \ldots, x_n\}$ then $\left(A, \|\cdot\|_p\right)$ is Banach space.*

Proof. We consider any Cauchy sequence (y_k) in A, writing $y_k = \alpha_{k1}x_1 + \alpha_{k2}x_2 + \ldots + \alpha_{kn}x_n$. Since (y_k) is Cauchy sequence, we have for every $\epsilon > 0$ there exist N_ϵ such that

$$\begin{aligned} \|y_k - y_l\|_p &= \left\| \sum_{i=1}^{n} (\alpha_{ki} - \alpha_{li})\, x_i \right\|_p \\ &= \left(\sum_{j=1}^{\infty} \left| \sum_{i=1}^{n} (\alpha_{ki} - \alpha_{li})\, \zeta_{ij} \right|^p \right)^{\frac{1}{p}} < \epsilon \end{aligned}$$

for all $k, l > N_\epsilon$. It follows that for every $j = 1, 2, \ldots$ we have

$$\left| \sum_{i=1}^{n} (\alpha_{ki} - \alpha_{li})\, \zeta_{ij} \right| < \epsilon_1.$$

Case 1. For some $i = 1, \ldots, n$, $\text{sign}\left(\sum_{i=1}^{n} (\alpha_{ki} - \alpha_{li})\, \zeta_{ij} \right) = \text{sign}\,(\alpha_{ki} - \alpha_{li})\, \zeta_{ij}$ holds. By Lemma 1, we can find $t \in (0,1)$ such that $\left| t\,(\alpha_{ki} - \alpha_{li})\, \zeta_{ij} \right| < \epsilon_1$.
Case 2. For some $i = 1, \cdots, n$, $\text{sign}\left(\sum_{i=1}^{n} (\alpha_{ki} - \alpha_{li})\, \zeta_{ij} \right) = \text{sign}\,(\alpha_{ki} - \alpha_{li})\, \zeta_{ij}$ holds. By Lemma 1, we can find $t \in (0,1)$ such that $\left| t\,(\alpha_{ki} - \alpha_{li})\, \zeta_{ij} \right| < \epsilon_1$.
So, we obtain $\left| t\zeta_{ij} \right| |(\alpha_{ki} - \alpha_{li})| < \epsilon_1$ for all $k, l \geq N_\epsilon$. Thus, for each fixed $i \in \{1, \ldots, n\}$, (α_{ki}) is a Cauchy sequence of real numbers. Hence, it is convergent, say $\alpha_{ki} \to \alpha_i$ as $k \to \infty$. Now, we can view that $\alpha = (\alpha_1, \ldots, \alpha_n)$ and we define $y := \alpha_1 x_1 + \ldots + \alpha_n x_n$. Its obvious that $y \in A$. Because α_k converge to α then y_k converge y. Hence, $\left(A, \|\cdot\|_p\right)$ is Banach Space. □

Next, we discuss a new inner product in ℓ^p and prove that A with the new inner product is Hilbert space. Write $b = \sum_{i=1}^{n} |x_i|$. We define a following mapping

$$\langle y, z \rangle_{b,2} := \sum_{k=1}^{\infty} b_k^{p-2} y_k z_k \tag{2.1}$$

for every $y, z \in \ell^p$. Using Hölder's inequality, we have

$$\sum_{k=1}^{\infty} b_k^{p-2} y_k z_k \leq \left[\sum_{k=1}^{\infty} b_k^p\right]^{\frac{p-2}{p}} \left[\sum_{k=1}^{\infty} y_k^p\right]^{\frac{1}{p}} \left[\sum_{k=1}^{\infty} z_k^p\right]^{\frac{1}{p}}.$$

Hence, the mapping $\langle \cdot, \cdot \rangle_{b,2}$ is defined on ℓ^p. Next, we have the following proposition.

Proposition 2.3. *The mapping* $\langle \cdot, \cdot \rangle_{b,2}$ *in* (2.1) *defines an inner product on* ℓ^p.

Proof. For every $y, z, w \in \ell^p$, we will verify that $\langle \cdot, \cdot \rangle_{b,2}$ satisfies the four properties of an inner product.

1) Since $\sum\limits_{k=1}^{\infty} b_k^{p-2} y_k^2 \geq 0$, we have $\langle y, y \rangle_{b,2} \geq 0$.

Next, we show that $\langle y, y \rangle_{b,2} = 0$ if only if $y = 0$. Suppose that $y = 0$, then

$$\langle 0, 0 \rangle_{b,2} = \sum_{k=1}^{\infty} b_k^{p-2} 0 = 0.$$

Conversely, if $\langle y, y \rangle_{b,2} = 0$, then $\sum\limits_{k=1}^{\infty} b_k^{p-2} y_k^2 = 0$. Since $b_k^{p-2} \neq 0$, we obtain $y = 0$.

2) Observe that

$$\langle y, z \rangle_{b,2} = \sum_{k=1}^{\infty} b_k^{p-2} y_k z_k = \sum_{k=1}^{\infty} b_k^{p-2} z_k y_k = \langle y, z \rangle_{b,2}.$$

3) Observe that

$$\langle \alpha y, z \rangle_{b,2} = \sum_{k=1}^{\infty} b_k^{p-2} \alpha y_k z_k = \alpha \langle y, z \rangle_{b,2}.$$

4) Observe that

$$\begin{aligned} \langle y, z + w \rangle_{b,2} &= \sum_{k=1}^{\infty} b_k^{p-2} y_k (z_k + w_k) \\ &= \sum_{k=1}^{\infty} b_k^{p-2} y_k z_k + \sum_{k=1}^{\infty} b_k^{p-2} y_k w_k \\ &= \langle y, z \rangle_{b,2} + \langle y, w \rangle_{b,2}. \end{aligned}$$

Therefore $\langle \cdot, \cdot \rangle_{b,2}$ defines an inner product on ℓ^p.

□

Corollary 2.4. *The following function*

$$\|y\|_{b,2} = \left[\sum_{k=1}^{\infty} b_k^{p-2} y_k^2\right]^{\frac{1}{2}} \tag{2.2}$$

defines a norm that corresponds to the inner product $\langle \cdot, \cdot \rangle_{b,2}$ *on* ℓ^p.

Using a norm that corresponds to the inner product $\langle\cdot,\cdot\rangle_{b,2}$, we have the following result.

Theorem 2.5. *The space* $\left(A, \|\cdot\|_{b,2}\right)$ *is complete. Accordingly,* $\left(A, \langle\cdot,\cdot\rangle_{b,2}\right)$ *is a Hilbert space.*

Proof. We consider any Cauchy sequence (y_k) in A, writing $y_k = \alpha_{k1}x_1 + \ldots + \alpha_{kn}x_n$ and $\alpha_k = (\alpha_{k1}, \ldots, \alpha_{kn})$. Since y_k is Cauchy, for every $\epsilon > 0$ there is an N_ϵ such that for all $k, l > N_\epsilon$,

$$\begin{aligned} \|y_k - y_l\|_{b,2} &= \left\| \sum_{i=1}^{n} (\alpha_{ki} - \alpha_{li})\, x_i \right\|_{b,2} \\ &= \left[\sum_{j=1}^{\infty} \sum_{i=1}^{n} b_j^{p-2} \left((\alpha_{ki} - \alpha_{li})\, \zeta_{ij} \right)^2 \right]^{\frac{1}{2}} < \epsilon. \end{aligned}$$

It follows that for every $j = 1, 2, \ldots$ we have

$$b_j^{p-2} \sum_{i=1}^{n} \left(\zeta_{ij}\alpha_{ki} - \zeta_{ij}\alpha_{li} \right)^2 < \epsilon^2.$$

So, we obtain $\left|\zeta_{ij}\right| b_j^{\frac{p-2}{2}} \left|\alpha_{ki} - \alpha_{li}\right| < \epsilon$ for all $k, l \geq N_\epsilon$. Thus, for each fixed $i \in \{1, \ldots, n\}$, (α_{ki}) is a Cauchy sequence of real numbers. Hence, it is convergent, say $\alpha_{ki} \to \alpha_i$ as $k \to \infty$. Now, we can view that $\alpha = (\alpha_1, \ldots, \alpha_n)$ and we define $y := \alpha_1 x_1 + \ldots + \alpha_n x_n$. Its obvious that $y \in A$. Because α_k converge to α then y_k converge y. Hence, $\left(A, \|\cdot\|_{b,2}\right)$ is complete. □

2.2. Angle betweeen two subspaces on A

We know that $\left(A, \langle\cdot,\cdot\rangle_{b,2}\right)$ is a Hilbert space with $A = \text{span}\,\{x_1, \cdots, x_n\}$ and, as before, $\{x_1, \cdots, x_n\}$ is a linearly independent set on ℓ^p. Using the Gram-Schmidt process, we have an orthonormal set $\{y_1, \cdots, y_n\}$. As a consequence, $span\{x_1, \cdots, x_n\} = \text{span}\,\{y_1, \cdots, y_n\}$. For every $u, v \in A$, we can write $u = \sum_{i=1}^{n} c_i y_i$ and $v = \sum_{i=1}^{n} d_i y_i$ where $c_i, d_i \in \mathbb{R}$ for every $i = 1, \ldots, n$. Moreover,

$$\begin{aligned} \langle u, v\rangle_{b,2} &= \left\langle \sum_{i=1}^{n} c_i y_i, \sum_{i=1}^{n} d_i y_i \right\rangle_{b,2} \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} c_i d_j \left\langle y_i, y_j \right\rangle_{b,2}. \end{aligned}$$

Because $\left\langle y_i, y_j \right\rangle_{b,2} = 0$ for $i \neq j$ and $\langle y_i, y_i\rangle_{b,2} = \|y_i\|_{b,2}^2$ then

$$\begin{aligned} \langle u, v\rangle_{b,2} &= \sum_{i=1}^{n} c_i d_i \|y_i\|_{b,2}^2 \\ &= \sum_{i=1}^{n} c_i d_i = \langle c, d\rangle, \end{aligned}$$

with $c = (c_1, \cdots, c_n)$ and $d = (d_1, \cdots, d_n)$. If $u = v$, then $\|u\|_{b,2} = \left(\sum_{i=1}^{n} c_i^2 \right)^{\frac{1}{2}}$.

According to the above form, we conclude that an inner product on A can be viewed as the inner product of the Euclidean space $\mathbb{R}^n$. Hence, an explisit formula of the angle between u and v of A, denoted θ, is given by

$$\cos\theta = \frac{\sum_{i=1}^{n} c_i d_i}{\left(\sum_{i=1}^{n} c_i^2 \sum_{i=1}^{n} d_i^2\right)^{\frac{1}{2}}}.$$

Note that the angle between two vectors of A is also the angle between two lines of A. Moreover, the angle between two vectors of A coincides with the angle between two vectors of $\mathbb{R}^n$.

Example 2.6. Let $A = \text{span}\,\{x_1, \cdots, x_5\}$ with $\{x_1, \cdots, x_5\}$ be the orthonormal set on ℓ^p. If $u = x_1 + 2x_2 + 3x_4 + 4x_5$ and $v = x_2 + 3x_3 + 5x_4 + 2x_5$ then the angle between u and v of A is

$$\cos\theta = \frac{25}{(30.39)^{\frac{1}{2}}} = \frac{25}{34,2}.$$

Next, we can discuss angle between 1-dimensional subspace and $(n-1)$-dimensional subspace where $n \geq 2$ of A. The result is shown as follows.

Proposition 2.7. *If $U = span\{x_k\}$ where $k = 1, \ldots, n$ and $V = span\{x_{i_2(k)}, \ldots, x_{i_n(k)}\}$ where $\{i_2(k), \ldots, i_n(k)\} = \{1, 2, \ldots, n\} - \{k\}$ of A , then the angle between subspaces U and V is θ $(0 \leq \theta \leq \frac{\pi}{2})$ with*

$$\cos^2\theta = \frac{1}{\|x_k\|_{b,2}^2} \sum_{j\in\{1,\cdots,n\}-\{k\}} \left\langle x_k, y_j \right\rangle_{b,2}^2.$$

Proof. Without loss of generality, write $k = 1$. Using the Gram-Schmidt process, we have an orthonormal set $\{y_2, \cdots, y_n\}$. As a consequence, span $\{x_2, \cdots, x_n\}$ = span $\{y_2, \cdots, y_n\}$. The projection of x_1 on V is given by

$$x_1^V = \sum_{j=2}^{n} \left\langle x_1, y_j \right\rangle_{b,2} y_j.$$

Then, we have

$$\begin{aligned} \left\langle x_1^V, x_1^V \right\rangle_{b,2} &= \left\langle \sum_{j=2}^{n} \left\langle x_1, y_j \right\rangle_{b,2} y_j, \sum_{j=2}^{n} \left\langle x_1, y_j \right\rangle_{b,2} y_j \right\rangle_{b,2} \\ &= \sum_{j=2}^{n} \sum_{i=2}^{n} \left\langle x_1, y_j \right\rangle_{b,2} \langle x_1, y_i \rangle_{b,2} \left\langle y_j, y_i \right\rangle_{b,2} \\ &= \sum_{j=2}^{n} \left\langle x_1, y_j \right\rangle_{b,2}^2. \end{aligned}$$

Hence, we obtain

$$\cos^2\theta = \frac{\|x_1^V\|_{b,2}^2}{\|x_1\|_{b,2}^2} = \frac{1}{\|x_1\|_{b,2}^2} \sum_{j=2}^{n} \left\langle x_1, y_j \right\rangle_{b,2}^2.$$

□

Example 2.8. Let $A = \text{span}\{x_1, x_2, x_3\}$ with $\{x_1, x_2, x_3\}$ be the orthonormal set in ℓ^1. Take $x_1 = (1, 0, 0, 0, \ldots)$, $x_2 = (0, 1, 0, 0, \ldots)$ and $x_3 = (0, 0, 1, 0, \ldots)$, so that we have $b = (1, 1, 1, 0, \ldots)$. Clearly $\|x_1\|^2_{b,2} = 1$, $\langle x_1, x_2\rangle^2_{b,2} = 0$ and $\langle x_1, x_3\rangle^2_{b,2} = 0$. If $U = \text{span}\{x_1\}$ and $V = \text{span}\{x_2, x_3\}$ of A then angle between U and V is θ with

$$\begin{aligned}\cos^2\theta &= \frac{1}{\|x_1\|^2_{b,2}}\left[\langle x_1, x_2\rangle^2_{b,2} + \langle x_1, x_3\rangle^2_{b,2}\right] \\ &= 0.\end{aligned}$$

Hence $\theta = \frac{\pi}{2}$.

Before we can discuss an explicit formula for the cosine of the angle between two subspaces of $A = \text{span}\{x_1, \cdots, x_n\}$, we recall definition of angle two subspace in inner product space as follows.

Definition 2.9. [3] Let $(X, \langle\cdot,\cdot\rangle)$ be a real inner product space. If $U = \text{span}\{u_1, \ldots, u_t\}$ is a t-dimensional subspace and $V = \text{span}\{v_1, \ldots, v_s\}$ is a s-dimensional subspace of X with $t \le s$, then the angle between subspaces U and V is defined by θ $(0 \le \theta \le \frac{\pi}{2})$ and

$$\cos^2\theta = \frac{\left\|u_1^V, \ldots, u_t^V\right\|^2}{\|u_1, \ldots, u_t\|^2},$$

where u_i^V denote the projection of u_i on V for each $i = 1, \ldots, t$.

Using Definition 2.9, we have the following theorem.

Theorem 2.10. *If $U = \text{span}\{x_1, \ldots, x_{n_1}\}$ is a n_1-dimensional subspace and $V = \text{span}\{x_{n_1+1}, \ldots, x_n\}$ is a $(n-n_1)$-dimensional subspace of A with $n_1 \le \frac{n}{2}$, then the angle between subspaces U and V is defined by θ and*

$$\cos^2\theta = \frac{\det\left([\langle x_i, y_l\rangle_{b,2}][\langle x_i, y_l\rangle_{b,2}]^T\right)}{\det\left([\left\langle x_i, x_j\right\rangle_{b,2}]\right)}$$

where $[\langle x_i, y_l\rangle_{b,2}]$ is a $(n_1 \times (n-n_1))$ matrix and $[\langle x_i, y_l\rangle_{b,2}]^T$ is its transpose for $i, j = 1, \ldots, n_1$.

Proof. Suppose that $\{x_{n_1+1}, \ldots, x_n\}$ is linearly independent. Using the Gram-Schmidt process, we obtain the orthonormal set $\{y_{n_1+1}, \ldots, y_n\}$. Here $\text{span}\{x_{n_1+1}, \ldots, x_n\} = \text{span}\{y_{n_1+1}, \ldots, y_n\}$. For each $i = 1, \ldots n_1$, the projection of x_i on V is given by

$$x_i^V = \sum_{l=n_1+1}^{n} \langle x_i, y_l\rangle_{b,2}\, y_j.$$

So, for $i, j = 1, \ldots, n_1$, we have

$$\begin{aligned}\left\langle x_i^V, x_j^V\right\rangle_{b,2} &= \left\langle x_i, x_j^V\right\rangle_{b,2} = \left\langle x_i, \sum_{l=n_1+1}^{n}\left\langle x_j, y_l\right\rangle_{b,2} y_l\right\rangle_{b,2} \\ &= \sum_{l=n_1+1}^{n}\langle x_i, y_l\rangle_{b,2}\left\langle x_j, y_l\right\rangle_{b,2}.\end{aligned}$$

Next, using formula angle in [3], we obtain

$$\left\|x_1^V, \ldots, x_{n_1}^V\right\|^2 = \det\left[\sum_{l=1}^{n_1} \langle x_i, y_l\rangle_{b,2} \left\langle x_j, y_l\right\rangle_{b,2}\right] = \det\left([\langle x_i, y_l\rangle_{b,2}][\langle x_i, y_l\rangle_{b,2}]^T\right),$$

where $[\langle x_i, y_l\rangle_{b,2}]$ is a $(n_1 \times (n - n_1))$ matrix and $[\langle x_i, y_l\rangle_{b,2}]^T$ is its transpose. Therefore, cosine of the angle between U and V is

$$\cos^2\theta = \frac{\det\left([\langle x_i, y_l\rangle_{b,2}][\langle x_i, y_l\rangle_{b,2}]^T\right)}{\det\left([\left\langle x_i, x_j\right\rangle_{b,2}]\right)},$$

where $[\left\langle x_i, x_j\right\rangle_{b,2}]$ is a $(n_1 \times n_1)$ matrix.

□

Next, we discuss angle between two subspaces that intersects on a subspace of A. Write $A = \operatorname{span}\{x_1, \ldots, x_{n_1}, x_{n_1+1}, \ldots, x_{n_1+n_2}, x_{n_1+n_2+1}, \ldots, x_{n_1+n_2+n_3}\}$ with $n = n_1 + n_2 + n_3$. Suppose now that $U = \operatorname{span}\{x_1, \ldots, x_{n_1}, x_{n_1+1}, \ldots, x_{n_1+n_2}\}$ and $V = \operatorname{span}\{x_1, \ldots, x_{n_1}, x_{n_1+n_2+1}, \ldots, x_{n_1+n_2+n_3}\}$. We observe that U and V are subspace on $A \subseteq \ell^p$. Moreover, using Definition 2.9, we have angle between U and V as follows.

Theorem 2.11. *If $U = span\{x_1, \cdots, x_{n_1}, x_{n_1+1}, \cdots, x_{n_1+n_2}\}$ is a $(n_1 + n_2)$-dimensional subspace and $V = span\{x_1, \cdots, x_{n_1}, x_{n_1+n_2+1}, \cdots, x_{n_1+n_2+n_3}\}$ is a $(n_1 + n_3)$-dimensional subspace of A with $n_2 \leq n_3$ that intersects on n_1-dimensional subspace $W = span\{x_1, \cdots, x_{n_1}\}$ then the angle between subspaces U and V is defined by θ with*

$$\cos^2\theta = \frac{\det\left[\left\langle (x_i^V)_W^\perp, (x_j^V)_W^\perp\right\rangle_{b,2}\right]}{\det\left[\left\langle (x_i)_W^\perp, (x_j)_W^\perp\right\rangle_{b,2}\right]},$$

where $(x_i^V)_W^\perp$, $(x_j^V)_W^\perp$, $(x_i)_W^\perp$ and $(x_j)_W^\perp$ are the orthogonal complement of u_i^V, u_j^V, u_i and u_j, respectively, on W for $i, j = n_1 + 1, \ldots, n_1 + n_2$.

Proof. The projection of x_i on V is x_i^V. Next, we may write $x_i^V = (x_i^V)_W + (x_i^V)_W^\perp$ where $(x_i^V)_W$ is the projection of x_i^V on W and $(x_i^V)_W^\perp$ is the orthogonal complement of x_i^V on W. In line with this, we may write $x_i = (x_i)_W + (x_i)_W^\perp$ where $(x_i)_W$ is the projection of x_i on W and $(x_i)_W^\perp$ is the orthogonal complement of x_i on W for $i = n_1 + 1, \ldots, n_1 + n_2$. Using the standard $(n_1 + n_2)$-norm and properties of determinans, we obtain

$$\begin{aligned}
\cos^2\theta &= \frac{\left\|x_1, \ldots, x_{n_1}, x_{n_1+1}^V, \ldots, x_{n_1+n_2}^V\right\|^2}{\left\|x_1, \ldots, x_{n_1}, x_{n_1+1}, \ldots, x_{n_1+n_2}\right\|^2} \\
&= \frac{\left\|x_1, \ldots, x_{n_1}, (x_{n_1+1}^V)_W + (x_{n_1+1}^V)_W^\perp, \ldots, (x_{n_1+n_2}^V)_W + (x_{n_1+n_2}^V)_W^\perp\right\|^2}{\left\|x_1, \ldots, x_{n_1}, (x_{n_1+1})_W + (x_{n_1+1})_W^\perp, \ldots, (x_{n_1+n_2})_W + (x_{n_1+n_2})_W^\perp\right\|^2} \\
&= \frac{\left\|x_1, \ldots, x_{n_1}\right\|^2 \left\|(x_{n_1+1}^V)_W^\perp, \ldots, (x_{n_1+n_2}^V)_W^\perp\right\|^2}{\left\|x_1, \ldots, x_{n_1}\right\|_{n_1}^2 \left\|(x_{n_1+1})_W^\perp, \ldots, (x_{n_1+n_2})_W^\perp\right\|^2}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\left\|(x^V_{n_1+1})^\perp_W, \ldots, (x^V_{n_1+n_2})^\perp_W\right\|^2}{\left\|(x_{n_1+1})^\perp_W, \ldots, (x_{n_1+n_2})^\perp_W\right\|^2} \\
&= \frac{\det\left[\left\langle (x^V_i)^\perp_W, (x^V_j)^\perp_W \right\rangle_{b,2}\right]}{\det\left[\left\langle (x_i)^\perp_W, (x_j)^\perp_W \right\rangle_{b,2}\right]}.
\end{aligned}
$$

with $[\left\langle (x^V_i)^\perp_W, (x^V_j)^\perp_W \right\rangle_{b,2}]$ and $[\left\langle (x_i)^\perp_W, (x_j)^\perp_W \right\rangle_{b,2}]$ are $(n_2 \times n_2)$ matrix. □

3. Conclusions

Based result has been given on the Sections 2, we have known that A that equipped with usual norm on ℓ^p is complete. We have introduced $\langle \cdot, \cdot \rangle_{b,2}$ on A and have shown that $\left(A, \|\cdot\|_{b,2}\right)$ is complete. Next, we have got angle between two vectors and between 1-dimensional subspace and $(s-1)$-dimensional subspace where $s \geq 2$ of A. Moreover, we have got angle between two subspaces that intersects on a subspace of A.

Acknowledgments

The first author is supported by PDUPT Program 2021 No. 752/UN4.22/PT.01.03/2021.

Conflict of interest

The authors declare that there is no conflict of interest.

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