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1 A boundary element method for anisotropic-diffusion convection-reaction equation in quadratically graded media of incompressible flow

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Abstract. A boundary element method (BEM) is utilized to find numerical solutions to boundary value problems of quadratically graded media governed by a spatially varying coefficients anisotropic-diffusion convection-reaction (DCR) equation. The variable coefficients equation is firstly transformed into a constant coefficients equation for which a boundary integral equation can be formulated. A BEM is then derived from the boundary integral equation. Some problems are considered. A FORTRAN script is developed for the computation of the solutions. The numerical solutions verify the validity of the analysis used to derive the boundary element method with accurate and consistent solutions. The computation shows that the BEM procedure elapses very efficient time in producing the solutions. In addition, results obtained for the considered examples show the effect of anisotropy and inhomogeneity of the media on the solutions. An example of a layered material is presented as an illustration of the application.

1. Introduction

By referring to the two-dimensional Cartesian coordinate system Ox_1x_2 this paper will concern with the DCR equation of incompressible flow

$$\frac{\partial}{\partial x_i} \left[d_{ij}(\mathbf{x}) \frac{\partial \beta(\mathbf{x})}{\partial x_j} \right] - \nu_i(\mathbf{x}) \frac{\partial \beta(\mathbf{x})}{\partial x_i} - \rho(\mathbf{x}) \beta(\mathbf{x}) = 0 \quad (1)$$

where $i, j = 1, 2$, $\mathbf{x} = (x_1, x_2)$, d_{ij} is the anisotropic diffusion or conduction coefficient, ν_i is the velocity, ρ is the reaction coefficient and β is the dependent variable. Within



the domain in question $[d_{ij}]$ is a real symmetrical matrix satisfying $d_{11}d_{22} - d_{12}^2 > 0$. For the repeated indices in equation (1) summation convention applies so that equation (1) can be written explicitly

$$\frac{\partial}{\partial x_1} \left(d_{11} \frac{\partial \beta}{\partial x_1} \right) + \frac{\partial}{\partial x_1} \left(d_{12} \frac{\partial \beta}{\partial x_2} \right) + \frac{\partial}{\partial x_2} \left(d_{12} \frac{\partial \beta}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left(d_{22} \frac{\partial \beta}{\partial x_2} \right) - \nu_1 \frac{\partial \beta}{\partial x_1} - \nu_2 \frac{\partial \beta}{\partial x_2} - \rho \beta = 0$$

Also, in equation (1) the coefficients d_{ij} , ν_i and ρ vary spatially and continuously which is applicable when the medium is inhomogeneous or is a functionally graded material.

DCR equation is usually used for modeling heat transfer and mass transport problems. According to Ravnik and Škerget [1], in mass transport which frequently occurs in environments, the convection process takes place with a flow velocity which varies in the medium in question, and in the case of turbulence modeling with turbulent viscosity hypothesis, the diffusivity also changes in the domain. These situations imply the DCR equation (1) becomes relevant.

Functionally graded materials (FGMs) are materials possessing characteristics which vary (with time and position) according to a mathematical function. FGMs are mainly artificial materials which are produced to meet a preset practical performance (see for example [2, 3]). Heat transfer in FGMs, for which equation (1) is usually used as the governing equation, is among applications that has been considerably studied by many people. This constitutes the relevancy of solving equation (1).

A number of papers previously considering DCR equation are [4, 5] in which AL-Bayati and Wrobel solved an isotropic-DCR equation with variable velocity, [6] where AL-Bayati and Wrobel solved an isotropic-DCR equation with a source term, [7] in which Fendoğlu et al. considered a constant coefficients unsteady isotropic-DCR equation with a source term, [8] by Rocca et al. which concerned with an isotropic-DCR equation with variable velocity, [9] in which Samec and Škerget solved an isotropic-DCR equation with variable velocity.

Not so many works have been done on DCR equation of type (1) for anisotropic FGMs where the diffusivity, velocity and reaction coefficients are simultaneously variable. Similar works of anisotropic materials but for different kinds of applications have been done previously in [10, 11, 12, 13] for Helmholtz equation, in [14, 15, 16] for DC equation, in [17, 18, 19, 20] for vector elliptic equation, in [21] for a scalar Laplace type equation, in [22, 23, 24, 25] for a scalar elliptic type equation, and in [26, 27, 28] for a modified Helmholtz type equation.

Numerical solutions β and its derivatives $\partial\beta/\partial x_1$, $\partial\beta/\partial x_2$ to (1) in the domain Ω , subjected to the boundary condition that either β or

$$P = d_{ij} (\partial\beta/\partial x_i) n_j \quad (2)$$

is known on the boundary $\partial\Omega$, are sought. The investigation of this paper is strictly mathematical. The purpose is mainly to develop a boundary element method for finding the numerical solutions.

2. Simplification to equation of constant coefficients

We limit the coefficients ν_i , d_{ij} and ρ to be varying spatially according to a specific continuous function $h(\mathbf{x})$

$$d_{ij}(\mathbf{x}) = \hat{d}_{ij} h(\mathbf{x}) \quad (3)$$

$$\nu_i(\mathbf{x}) = \hat{\nu}_i h(\mathbf{x}) \quad (4)$$

$$\rho(\mathbf{x}) = \hat{\rho} h(\mathbf{x}) \quad (5)$$

where \hat{d}_{ij} , $\hat{\nu}_i$ and $\hat{\rho}$ are constant and the inhomogeneity function $h(\mathbf{x})$ is a differentiable function taking the quadratic form

$$h(\mathbf{x}) = [A(\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2)]^2 \quad (6)$$

where A and α_i are constants. So the material under consideration is a quadratically graded material. With this quadratic inhomogeneity function the coefficients d_{ij} , ν_i and ρ can be decomposed into a constant and a variable velocity which was considered in papers [1, 9, 5, 4].

Substitution of (3), (4) and (5) into (1) gives

$$\hat{d}_{ij} \frac{\partial}{\partial x_i} \left(h \frac{\partial \beta}{\partial x_j} \right) - h \hat{\nu}_i \frac{\partial \beta}{\partial x_i} - \hat{\rho} h \beta = 0 \quad (7)$$

Write

$$\beta(\mathbf{x}) = h^{-1/2}(\mathbf{x}) \varsigma(\mathbf{x}) \quad (8)$$

then equation (7) can be rewritten as

$$\hat{d}_{ij} \frac{\partial}{\partial x_i} \left[h \frac{\partial (h^{-1/2} \varsigma)}{\partial x_j} \right] - h \hat{\nu}_i \frac{\partial (h^{-1/2} \varsigma)}{\partial x_i} - \hat{\rho} h^{1/2} \varsigma = 0$$

which can be further written as

$$\hat{d}_{ij} \left[\left(\frac{1}{4} h^{-3/2} \frac{\partial h}{\partial x_i} \frac{\partial h}{\partial x_j} - \frac{1}{2} h^{-1/2} \frac{\partial^2 h}{\partial x_i \partial x_j} \right) \varsigma + h^{1/2} \frac{\partial^2 \varsigma}{\partial x_i \partial x_j} \right] - h \hat{\nu}_i \frac{\partial (h^{-1/2} \varsigma)}{\partial x_i} - \hat{\rho} h^{1/2} \varsigma = 0 \quad (9)$$

The identities

$$\begin{aligned} \frac{\partial^2 h^{1/2}}{\partial x_i \partial x_j} &= - \left(\frac{1}{4} h^{-3/2} \frac{\partial h}{\partial x_i} \frac{\partial h}{\partial x_j} - \frac{1}{2} h^{-1/2} \frac{\partial^2 h}{\partial x_i \partial x_j} \right) \\ h \frac{\partial (h^{-1/2} \varsigma)}{\partial x_i} &= h^{1/2} \frac{\partial \varsigma}{\partial x_i} - \varsigma \frac{\partial h^{1/2}}{\partial x_i} \end{aligned}$$

allow equation (9) to be written in the form

$$h^{1/2} \left(\hat{d}_{ij} \frac{\partial^2 \varsigma}{\partial x_i \partial x_j} - \hat{\nu}_i \frac{\partial \varsigma}{\partial x_i} - \hat{\rho} \varsigma \right) - \varsigma \left(\hat{d}_{ij} \frac{\partial^2 h^{1/2}}{\partial x_i \partial x_j} - \hat{\nu}_i \frac{\partial h^{1/2}}{\partial x_i} \right) = 0 \quad (10)$$

For incompressible flows, substitution of (4) into the zero velocity divergence $\partial \nu_i(\mathbf{x})/\partial x_i = 0$ gives

$$\hat{\nu}_i \frac{\partial h^{1/2}}{\partial x_i} = 0$$

which implies

$$\hat{\nu}_i \alpha_i = 0 \quad (11)$$

Then the quadratic inhomogeneity function (6) satisfies

$$\hat{d}_{ij} \frac{\partial^2 h^{1/2}}{\partial x_i \partial x_j} - \hat{\nu}_i \frac{\partial h^{1/2}}{\partial x_i} = 0 \quad (12)$$

Therefore substitution of (12) into (10) gives a constant coefficients equation

$$\hat{d}_{ij} \frac{\partial^2 \zeta}{\partial x_i \partial x_j} - \hat{\nu}_i \frac{\partial \zeta}{\partial x_i} - \hat{\rho} \zeta = 0 \quad (13)$$

Furthermore use of (3) and (8) in (2) yields

$$P = -P_h \zeta + P_\zeta h^{1/2} \quad (14)$$

where $P_h(\mathbf{x}) = \hat{d}_{ij} (\partial h^{1/2} / \partial x_j) n_i$ and $P_\zeta(\mathbf{x}) = \hat{d}_{ij} (\partial \zeta / \partial x_j) n_i$.

3. The integral equation

By using Gauss divergence theorem, equation (13) can be transformed into a boundary integral equation

$$\kappa(\boldsymbol{\chi}) \zeta(\boldsymbol{\chi}) = \int_{\partial\Omega} \{P_\zeta(\mathbf{x}) \Lambda(\mathbf{x}, \boldsymbol{\chi}) - [P_\nu(\mathbf{x}) \Lambda(\mathbf{x}, \boldsymbol{\chi}) + \Theta(\mathbf{x}, \boldsymbol{\chi})] \zeta(\mathbf{x})\} ds(\mathbf{x}) \quad (15)$$

where $P_\nu(\mathbf{x}) = \hat{\nu}_i n_i(\mathbf{x})$ and $\boldsymbol{\chi} = (\chi_1, \chi_2)$, $\kappa = 0$ if $(\chi_1, \chi_2) \notin \Omega \cup \partial\Omega$, $\kappa = 1$ if (χ_1, χ_2) lies inside the domain Ω , $\kappa = \frac{1}{2}$ if (χ_1, χ_2) is on the boundary $\partial\Omega$ given that $\partial\Omega$ has a continuously turning tangent at (χ_1, χ_2) . In (15) the fundamental solution $\Lambda(\mathbf{x}, \boldsymbol{\chi})$ for equation (13) satisfies

$$\hat{d}_{ij} \frac{\partial^2 \Lambda(\mathbf{x}, \boldsymbol{\chi})}{\partial x_i \partial x_j} + \hat{\nu}_i \frac{\partial \Lambda(\mathbf{x}, \boldsymbol{\chi})}{\partial x_i} - \hat{\rho} \Lambda(\mathbf{x}, \boldsymbol{\chi}) = -\delta(\mathbf{x} - \boldsymbol{\chi})$$

and $\Theta(\mathbf{x}, \boldsymbol{\chi})$ satisfies

$$\Theta(\mathbf{x}, \boldsymbol{\chi}) = \hat{d}_{ij} \frac{\partial \Lambda(\mathbf{x}, \boldsymbol{\chi})}{\partial x_j} n_i$$

where δ is the Dirac delta function. For 2-D problems the function Λ is given as (see Azis [29])

$$\Lambda(\mathbf{x}, \boldsymbol{\chi}) = \frac{\ddot{\tau}}{2\pi D} \exp\left(-\frac{\dot{\nu} \cdot \dot{\mathbf{R}}}{2D}\right) K_0(\mu \dot{R})$$

where $\mu = \sqrt{(\hat{\nu}/2D)^2 + (\hat{\rho}/D)}$, $D = [\hat{d}_{11} + 2\hat{d}_{12}\hat{\tau} + \hat{d}_{22}(\hat{\tau}^2 + \hat{\tau}^2)]/2$, $\hat{\mathbf{R}} = \dot{\mathbf{x}} - \dot{\mathbf{x}}$, $\hat{\mathbf{x}} = (x_1 + \hat{\tau}x_2, \hat{\tau}x_2)$, $\hat{\mathbf{x}} = (\chi_1 + \hat{\tau}\chi_2, \hat{\tau}\chi_2)$, $\hat{\nu} = (\hat{\nu}_1 + \hat{\tau}\hat{\nu}_2, \hat{\tau}\hat{\nu}_2)$, $\hat{R} = \sqrt{(x_1 + \hat{\tau}x_2 - \chi_1 - \hat{\tau}\chi_2)^2 + (\hat{\tau}x_2 - \hat{\tau}\chi_2)^2}$, and $\hat{\nu} = \sqrt{(\hat{\nu}_1 + \hat{\tau}\hat{\nu}_2)^2 + (\hat{\tau}\hat{\nu}_2)^2}$ where $\hat{\tau}$ and $\hat{\tau}$ are respectively the real and the positive imaginary parts of the complex root τ of the quadratic equation $\hat{d}_{11} + 2\hat{d}_{12}\tau + \hat{d}_{22}\tau^2 = 0$ and K_0 is the modified Bessel function. Use of (8) and (14) in (15) yields

$$\kappa h^{1/2} \beta = \int_{\partial\Omega} \left\{ (h^{-1/2} \Lambda) P + \left[(P_h - P_\nu h^{1/2}) \Lambda - h^{1/2} \Theta \right] \beta \right\} ds \quad (16)$$

4. Discretisation

Divide the boundary $\partial\Omega$ into L segments $\partial\Omega_l \equiv [\mathbf{q}_{l-1}, \mathbf{q}_l]$ for $l = 1, 2, 3, \dots, L$ where \mathbf{q}_{l-1} and \mathbf{q}_l are the endpoints of the segment $\partial\Omega_l$. It is assumed that β and P are constant along each boundary segment $\partial\Omega_l$ taking on their values at the mid point $\hat{\mathbf{q}}_l = (\mathbf{q}_{l-1} + \mathbf{q}_l)/2$. Then the discretised form of (16) may be written as

$$\begin{aligned} \kappa(\boldsymbol{\chi}) h^{1/2}(\boldsymbol{\chi}) \beta(\boldsymbol{\chi}) &= \sum_{l=1}^L \left\{ P(\hat{\mathbf{q}}_l) \int_{\mathbf{q}_{l-1}}^{\mathbf{q}_l} \left[h^{-1/2}(\mathbf{x}) \Lambda(\mathbf{x}, \boldsymbol{\chi}) \right] ds(\mathbf{x}) \right. \\ &+ \beta(\hat{\mathbf{q}}_l) \int_{\mathbf{q}_{l-1}}^{\mathbf{q}_l} \left\{ \left[P_h(\mathbf{x}) - P_\nu(\mathbf{x}) h^{1/2}(\mathbf{x}) \right] \Lambda(\mathbf{x}, \boldsymbol{\chi}) \right. \\ &\left. \left. - h^{1/2}(\mathbf{x}) \Theta(\mathbf{x}, \boldsymbol{\chi}) \right\} ds(\mathbf{x}) \right\} \end{aligned} \quad (17)$$

The integral equation (17) is used to find the boundary unknowns $\beta(\mathbf{x})$ or $P(\mathbf{x})$ on $\partial\Omega$. Then the solutions $\beta(\mathbf{x})$ and its derivatives in the domain Ω are evaluated using the complete boundary data $\beta(\mathbf{x})$ and $P(\mathbf{x})$ on $\partial\Omega$.

If the source point $\boldsymbol{\chi}$ lies on the boundary $\partial\Omega$ (thus $\eta(\boldsymbol{\chi}) = \frac{1}{2}$), say $\boldsymbol{\chi}$ lies on the boundary segment $\partial\Omega_k$ ($k = 1, 2, \dots, L$) so that $\boldsymbol{\chi}$ is the mid-point $\hat{\mathbf{q}}_k$ of $\partial\Omega_k$, then equation (17) can be written as

$$\begin{aligned} \frac{1}{2} h^{1/2}(\hat{\mathbf{q}}_k) \beta(\hat{\mathbf{q}}_k) &= \sum_{l=1}^L \left\{ P(\hat{\mathbf{q}}_l) \int_{\mathbf{q}_{l-1}}^{\mathbf{q}_l} \left[h^{-1/2}(\mathbf{x}) \Lambda(\mathbf{x}, \hat{\mathbf{q}}_k) \right] ds(\mathbf{x}) \right. \\ &+ \beta(\hat{\mathbf{q}}_l) \int_{\mathbf{q}_{l-1}}^{\mathbf{q}_l} \left\{ \left[P_h(\mathbf{x}) - P_\nu(\mathbf{x}) h^{1/2}(\mathbf{x}) \right] \Lambda(\mathbf{x}, \hat{\mathbf{q}}_k) \right. \\ &\left. \left. - h^{1/2}(\mathbf{x}) \Theta(\mathbf{x}, \hat{\mathbf{q}}_k) \right\} ds(\mathbf{x}) \right\} \end{aligned}$$

for $k = 1, 2, \dots, L$. This equation may be written in matrix form

$$\frac{1}{2} h_k^{1/2} \beta_k - \sum_{l=1}^L \hat{H}_{kl} \beta_l = \sum_{l=1}^L G_{kl} P_l \quad (18)$$

where $h_k^{1/2} = h^{1/2}(\dot{\mathbf{q}}_k)$, $\beta_k = \beta(\dot{\mathbf{q}}_k)$, $P_l = P(\dot{\mathbf{q}}_l)$, and

$$\hat{H}_{kl} = \int_{\mathbf{q}_{l-1}}^{\mathbf{q}_l} \left\{ \left[P_h(\mathbf{x}) - P_\nu(\mathbf{x}) h^{1/2}(\mathbf{x}) \right] \Lambda(\mathbf{x}, \dot{\mathbf{q}}_k) - h^{1/2}(\mathbf{x}) \Theta(\mathbf{x}, \dot{\mathbf{q}}_k) \right\} ds(\mathbf{x}) \quad (19)$$

$$G_{kl} = \int_{\mathbf{q}_{l-1}}^{\mathbf{q}_l} \left[h^{-1/2}(\mathbf{x}) \Lambda(\mathbf{x}, \dot{\mathbf{q}}_k) \right] ds(\mathbf{x}) \quad (20)$$

The integrals in equations (19) and (20) can be evaluated numerically. And also the values of the modified Bessel function involved in the fundamental solutions Λ and Θ can be approached by their polynomial approximations (see Abramowitz and Stegun [30]). In a simpler way, equation (18) may be written as

$$\sum_{l=1}^L H_{kl} \beta_l = \sum_{l=1}^L G_{kl} P_l \quad (21)$$

where

$$H_{kl} = \begin{cases} -\hat{H}_{kl} & \text{when } k \neq l \\ \frac{1}{2} h_k^{1/2} - \hat{H}_{kl} & \text{when } k = l \end{cases}$$

Equation (21) can be rearranged by putting the unknowns on the left hand side and all the knowns on the right hand side to obtain a $L \times L$ linear system of algebraic equations in the form

$$\mathbf{A}\mathbf{X} = \mathbf{B} \quad (22)$$

where \mathbf{X} is the unknown matrix.

Once the unknowns β and P on the boundary $\partial\Omega$ are obtained from the equation (22), we can calculate the value of β at any point χ inside the domain Ω by using the following equation

$$\beta(\chi) = h^{-1/2}(\chi) \sum_{l=1}^L \left\{ P(\dot{\mathbf{q}}_l) \int_{\mathbf{q}_{l-1}}^{\mathbf{q}_l} \left[h^{-1/2}(\mathbf{x}) \Lambda(\mathbf{x}, \chi) \right] ds(\mathbf{x}) + \beta(\dot{\mathbf{q}}_l) \int_{\mathbf{q}_{l-1}}^{\mathbf{q}_l} \left\{ \left[P_h(\mathbf{x}) - P_\nu(\mathbf{x}) h^{1/2}(\mathbf{x}) \right] \Lambda(\mathbf{x}, \chi) - h^{1/2}(\mathbf{x}) \Theta(\mathbf{x}, \chi) \right\} ds(\mathbf{x}) \right\}$$

We can also calculate the values of the derivatives $\partial\beta/\partial\chi_1$ and $\partial\beta/\partial\chi_2$ using the following equations

$$\begin{aligned} \frac{\partial\beta}{\partial\chi_1}(\chi) &= h^{-1/2}(\chi) \left\{ \sum_{l=1}^L \left\{ P(\dot{\mathbf{q}}_l) \int_{\mathbf{q}_{l-1}}^{\mathbf{q}_l} \left[h^{-1/2}(\mathbf{x}) \frac{\partial\Lambda(\mathbf{x}, \chi)}{\partial\chi_1} \right] ds(\mathbf{x}) + \beta(\dot{\mathbf{q}}_l) \int_{\mathbf{q}_{l-1}}^{\mathbf{q}_l} \left[\left[P_h(\mathbf{x}) - P_\nu(\mathbf{x}) h^{1/2}(\mathbf{x}) \right] \frac{\partial\Lambda(\mathbf{x}, \chi)}{\partial\chi_1} - h^{1/2}(\mathbf{x}) \frac{\partial\Theta(\mathbf{x}, \chi)}{\partial\chi_1} \right] ds(\mathbf{x}) \right\} - \beta(\chi) \frac{\partial h^{1/2}(\chi)}{\partial\chi_1} \right\} \end{aligned}$$

$$\frac{\partial \beta}{\partial \chi_2}(\chi) = h^{-1/2}(\chi) \left\{ \sum_{l=1}^L \left\{ P(\hat{q}_l) \int_{q_{l-1}}^{q_l} \left[h^{-1/2}(\mathbf{x}) \frac{\partial \Lambda(\mathbf{x}, \chi)}{\partial \chi_2} \right] ds(\mathbf{x}) \right. \right. \\ \left. \left. + \beta(\hat{q}_l) \int_{q_{l-1}}^{q_l} \left[[P_h(\mathbf{x}) - P_v(\mathbf{x}) h^{1/2}(\mathbf{x})] \frac{\partial \Lambda(\mathbf{x}, \chi)}{\partial \chi_2} \right. \right. \right. \\ \left. \left. \left. - h^{1/2}(\mathbf{x}) \frac{\partial \Theta(\mathbf{x}, \chi)}{\partial \chi_2} \right] ds(\mathbf{x}) \right\} - \beta(\chi) \frac{\partial h^{1/2}(\chi)}{\partial \chi_2} \right\}$$

5. Numerical examples

The aim of this section is to justify the analysis used to derive the boundary integral equation (16). Some problems will be considered. Solutions to the problems are calculated using a FORTRAN script and a specific command is embedded inside the script to count the elapsed CPU computation time for obtaining the results as to show the efficiency of the BEM. The other aspects that will be justified are the accuracy and consistency of the BEM solutions as to see whether or not the developed FORTRAN script works correctly. Moreover, we will also study the influence of the anisotropy and inhomogeneity of the material under consideration on the solutions.

5.1. A test problem

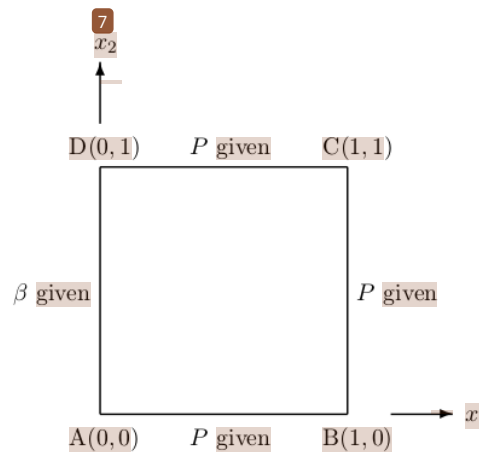


Figure 1. The domain Ω

The domain and the boundary conditions are as depicted in Figure 1. The constant coefficients are

$$\hat{d}_{ij} = \begin{bmatrix} 1 & 0.75 \\ 0.75 & 1.5 \end{bmatrix} \quad \hat{\nu}_i = (0.5, 0.3) \quad \hat{\rho} = 0.25$$

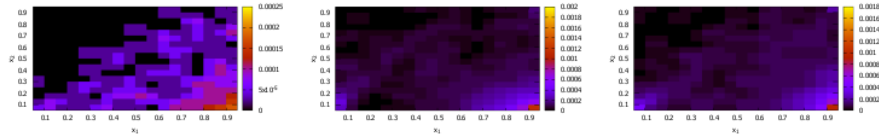


Figure 2. The absolute errors of β (left), $\partial\beta/\partial x_1$ (center), $\partial\beta/\partial x_2$ (right) solutions for the test problem

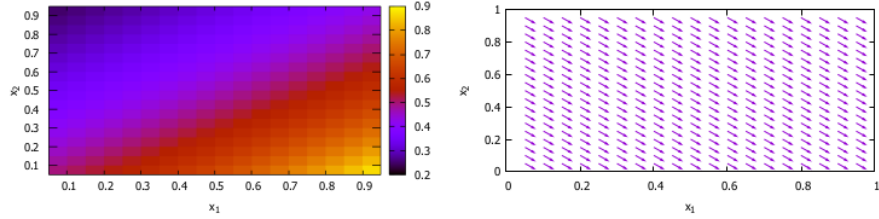


Figure 3. The BEM scattering β and flow vector $(\partial\beta/\partial x_1, \partial\beta/\partial x_2)$ solutions for the test problem

The quadratical inhomogeneity function h is assumed to be

$$h(\mathbf{x}) = [2(1 - 0.15x_1 + 0.25x_2)]^2$$

The exact solution is

$$\beta(\mathbf{x}) = 0.5 \exp(0.5x_1 - 0.584933x_2) / (1 - 0.15x_1 + 0.25x_2)$$

Figures 2 and 3 show the accuracy and consistency (between the scattering and flow) of the numerical solutions when a number of 320 boundary segments of equal length, namely 80 segments on each side, are used. Efficiency of the BEM is indicated by a short elapsed CPU time which is only in 185.703125 seconds for obtaining solutions $\beta(\mathbf{x})$ and its derivatives at 19×19 interior points.

5.2. Problems without any simple exact solutions

A layered material consisting of eight layers of the same size and with boundary conditions as depicted in Figure 4 is under consideration. Each layer is supposed to be homogeneous, but from layer to layer the diffusion d_{ij} , velocity ν_i and reaction ρ coefficients are assumed to be varying as smoothly as the variability can be fitted to a quadratical function $h(\mathbf{x}) = (\alpha_0 + \alpha_2 x_2)^2$.

Just as an illustration, suppose that we have a set of data of the diffusion d_{ij} , velocity ν_i and reaction ρ coefficient values at center point of each layer as shown in Table 1. And

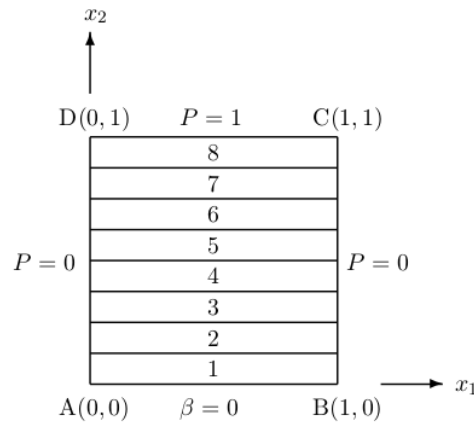


Figure 4. A layered material as the domain Ω

Table 1. Example of the diffusion d_{ij} , velocity ν_i and reaction ρ coefficient data

Layer	d_{11}	d_{12}	d_{22}	ν_1	ν_2	ρ
1	1.03785	0	1.55678	0.20757	0	0.25946
2	1.03785	0	1.67350	0.22313	0	0.27892
3	1.19629	0	1.79443	0.23926	0	0.29907
4	1.27973	0	1.91959	0.25595	0	0.31993
5	1.36598	0	2.04896	0.27320	0	0.341495
6	1.45504	0	2.18256	0.291015	0	0.36376
7	1.54691	0	2.32037	0.309385	0	0.38673
8	1.64160	0	2.46240	0.32832	0	0.410405

we also have reference values of constant coefficients

$$\hat{d}_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 1.5 \end{bmatrix} \quad \hat{\nu}_i = (0.2, 0) \quad \hat{\rho} = 0.25$$

Fitting the data in Table 1 to the function $h(\mathbf{x}) = (\alpha_0 + \alpha_2 x_2)^2$ we will get the values of the parameters α_0 and α_2

$$\alpha_0 = 1 \quad \alpha_2 = 0.3$$

Therefore we can then approximate the layered material as a sole material with continuously varying coefficients. So we may use the analysis in Sections 2 – 4 to solve the problem.

Again, a number of 320 segments of equal length are used. As shown in Figure 5 for the constant orthotropic diffusion coefficient \hat{d}_{ij} given above the solution β exhibits the nature of the considered medium as a layered material.

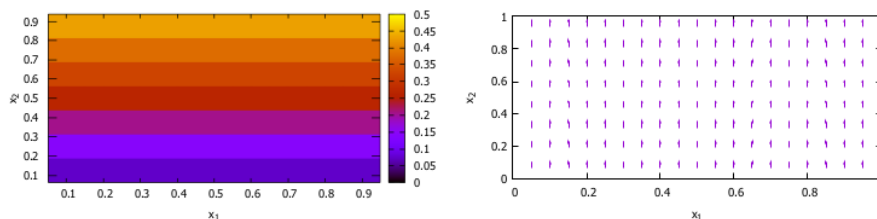


Figure 5. The BEM scattering β and flow vector $(\partial\beta/\partial x_1, \partial\beta/\partial x_2)$ solutions of the orthotropic inhomogeneous medium

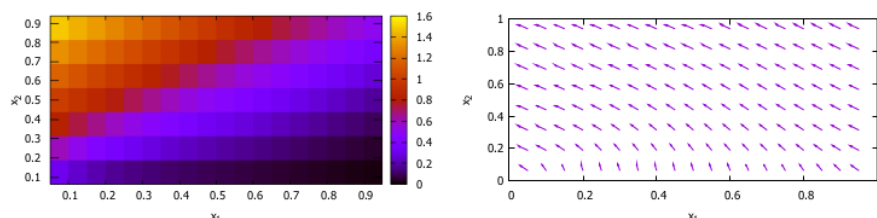


Figure 6. The BEM scattering β and flow vector $(\partial\beta/\partial x_1, \partial\beta/\partial x_2)$ solutions of the anisotropic inhomogeneous medium

However, if we assume that the material under consideration is a sole material varying continuously and we change the diffusion coefficient \hat{d}_{ij} to an anisotropic diffusion coefficient

$$\hat{d}_{ij} = \begin{bmatrix} 1 & 1 \\ 1 & 1.5 \end{bmatrix}$$

and keeping the other coefficients remain the same then we will obtain a significantly different solution β as shown in Figure 6. This means that the anisotropy of the medium gives an impact on the solution. Therefore in application it is necessary for the anisotropy to be taken into account.

Now if we assume that the medium is anisotropic homogeneous with

$$\hat{d}_{ij} = \begin{bmatrix} 1 & 1 \\ 1 & 1.5 \end{bmatrix} \quad \alpha_0 = 1 \quad \alpha_2 = 0$$

then we will obtain a slightly different solution β as shown in Figure 7. This indicates that the inhomogeneity also gives an impact on the solution. Therefore one should put the inhomogeneity in consideration for any application.

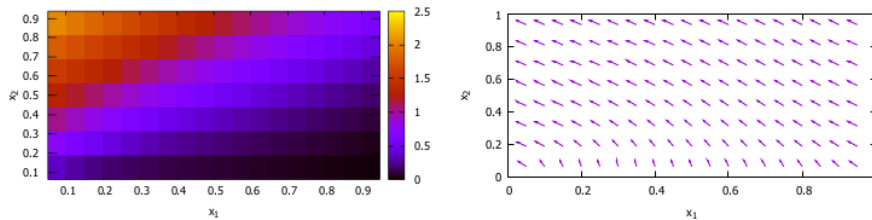


Figure 7. The BEM scattering β and flow vector $(\partial\beta/\partial x_1, \partial\beta/\partial x_2)$ solutions of the anisotropic homogeneous medium

6. Conclusion

A standard BEM has been used to find numerical solutions to boundary value problems governed by the anisotropic diffusion-convection-reaction equation (1) of incompressible flow with spatially variable coefficients for quadratically graded (inhomogeneous) media. The BEM gives accurate and consistent solution and uses very efficient computation time for producing the results, which verifies that the analysis for deriving the boundary integral equation in Section 3 is valid and the developed FORTRAN code works well. Moreover, effect of the anisotropy and inhomogeneity of the media on the solutions are also presented.

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